L^2-Betti numbers and kernels of maps to $\mathbb{Z}$

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Abstract

The main focus of this thesis is the connection between the L^2-Betti numbers of RFRS groups and the existence of epimorphisms to $\mathbb{Z}$ with kernels having certain desirable properties.

In particular, we show that if G is a RFRS group of type $\mathrm{FP}_n(\mathbb{Q})$ for some $n \geqslant 0$, then G has a finite-index subgroup $H \leqslant G$ admitting an epimorphism $H \to \mathbb{Z}$ with kernel of type $\mathrm{FP}_n(\mathbb{Q})$ if and only if $b_i^{(2)}(G) = 0$ for all $i \leqslant n$. A consequence is that the fundamental group of any closed hyperbolic manifold with cubulated fundamental group virtually algebraically fibres with kernel of type $\mathrm{FP}(\mathbb{Q})$.

We also prove that if G is a RFRS group of type $\mathrm{FP}(\mathbb{Q})$ and with $\mathrm{cd}_{\mathbb{Q}}(G) = n$, then G admits a virtual map to $\mathbb{Z}$ with kernel of rational cohomological dimension $n-1$ if and only if $b_n^{(2)}(G) = 0$. In particular, we show that a finitely generated RFRS group of cohomological dimension two is virtually free-by-cyclic if and only if its second L^2-Betti number vanishes (we stress that the free kernel of the free-by-cyclic group is finitely generated if and only if the first L^2-Betti number vanishes as well). We obtain more general results in the wider class of residually poly-$\mathbb{Z}$ groups. We also prove analogues of the results in this and the previous paragraph over fields of positive characteristic, where the L^2-Betti numbers must be replaced with suitable positive characteristic analogues.

Finally, and in a slightly different direction, we show that any group algebra of a torsion-free 3-manifold group embeds into a division ring, and as a consequence show that group algebras of torsion-free 3-manifold groups satisfy Kaplansky's Zero Divisor Conjecture.

Statement of originality

I declare that, to the best of my knowledge, the work in this thesis is entirely my own, except where otherwise stated or cited.

The material of Chapter 3 is based on the article [Fis24a] and on unpublished work with Giovanni Italiano and Dawid Kielak. The material of Chapter 4 is based on the preprint [Fis24b], the preprint [FK24] joint with Kevin Klinge, and on unpublished work with Pablo Sánchez-Peralta. The material of Chapter 5 is based on the article [FSP23], which is joint with Pablo Sánchez-Peralta.

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Chapter 1

Introduction

Virtually special groups, introduced by Haglund and Wise in [HW08], play a central role in geometric group theory and low-dimensional topology. A group G is *virtually (compact) special* if a finite-index subgroup $H \leqslant G$ acts properly (and cocompactly) on a CAT(0) cube complex X such that the hyperplanes of the quotient X/G avoid certain pathologies, which are outlined in [HW08] but will not be important here. The list of groups known to be virtually special is now extensive, and includes finitely generated free groups, surface groups, right-angled Artin groups, limit groups [Wis21], one-relator groups with torsion [Wis21], one-relator groups with negative immersions [Lin22], hyperbolic F_n-by-$\mathbb{Z}$ groups [HW10a], small cancellation groups [Wis04], Coxeter groups [HW10b], fundamental groups of finite-volume hyperbolic 3-manifolds [Ago13, Wis21], hyperbolic groups acting geometrically on CAT(0) cube complexes [Ago08], some relatively hyperbolic cubulated groups [Rey23], simple type lattices in $\mathrm{SO}(n, 1)$ [BHW11], and many more.

The example of finite-volume hyperbolic 3-manifold groups mentioned above is historically important, as it relates to Thurston's Virtual Fibring Conjecture, which predicted that every finite-volume hyperbolic 3-manifold admits a finite-sheeted cover that is a surface bundle over S^1. In [Ago08], Agol isolated the group-theoretic *residually finite rationally solvable* (or *RFRS*) property–which special groups possess–and showed if M is a compact, irreducible 3-manifold such that $\pi_1(M)$ is virtually RFRS, then M virtually fibres over the circle if and only if $\chi(M) = 0$. The Virtual Fibring Conjecture was resolved by Agol [Ago13] and Wise [Wis21], building on the previous work of Kahn and Marković [KM12], by showing that if M is a finite-volume hyperbolic 3-manifold, then $\pi_1(M)$ is virtually compact special, and therefore is virtually RFRS. The RFRS property will be discussed in Section 2.4, though for now we mention that a finitely generated group is RFRS if and only if it is residually (poly-$\mathbb{Z}$ and virtually Abelian).

1.1 Algebraic fibring

Recently, Kielak established a group-theoretic generalisation of Agol's fibring criterion [Kie20b]. Before stating it, we briefly discuss algebraic fibrations and L^2-invariants. A group G *algebraically fibres* if it admits an epimorphism $G \to \mathbb{Z}$ with finitely generated kernel. The terminology is motivated from topology: if a compact aspherical space X decomposes as a fibre bundle over S^1, then the induced map $\pi_1(X) \to \pi_1(S^1) \cong \mathbb{Z}$ is an algebraic fibration. The L^2-Betti numbers are numerical homological invariants that have proven to be powerful tools in topology and in the theory of infinite groups. The L^2-Betti numbers of a RFRS groups can be calculated as follows: if G is RFRS, then it follows from work of Schick [Sch02] that the group algebra $\mathbb{Q}[G]$ embeds into a division ring $\mathcal{D}_{\mathbb{Q}[G]}$ called the *Linnell division ring* of G. The homology module $\mathrm{H}_n(G; \mathcal{D}_{\mathbb{Q}[G]})$ is a module over $\mathcal{D}_{\mathbb{Q}[G]}$ and therefore has a well defined dimension, which is the nth L^2-Betti number of G, denoted by $b_n^{(2)}(G)$. Thanks to the work of [Lüc02, Theorem 7.2], it has long been known that the non-vanishing of $b_1^{(2)}(G)$ is an obstruction to G virtually algebraically fibring. Kielak's theorem provides a converse in the realm of RFRS groups.

Theorem 1.1.1 ([Kie20b, Theorem 5.3])**.** *If G is a non-trivial finitely generated RFRS group, then G virtually algebraically fibres if and only if $b_1^{(2)}(G) = 0$.*

In [Sta62], Stallings shows that if M is a 3-manifold for which there exists an algebraic fibration $\pi_1(M) \to \mathbb{Z}$, then this algebraic fibration is induced by a fibration of M over S^1 (the original statement of his result had some assumptions that can now be dropped thanks to geometrisation). Moreover, from [LL95] (see also Example A.8), $b_1^{(2)}(\pi_1(M)) = -\chi(M)$ for an aspherical, irreducible 3-manifold M. Thus, Kielak's theorem is an honest generalisation of Agol's fibring criterion.

The first main result of this thesis is a higher-degree generalisation of Theorem 1.1.1. By a theorem of Gaboriau [Gab02, Théorème 6.6], if G is a countable group such that there is an extension $1 \to K \to G \to Q \to 1$ with Q infinite amenable and $b_n^{(2)}(K) < \infty$ for some $n \geqslant 0$, then $b_n^{(2)}(G) = 0$. Recall that a group G is of *type* $\mathrm{FP}_n(R)$ for some ring R if the trivial $R[G]$-module R admits a partial resolution of length n by finitely generated projective $R[G]$-modules; this is a homological analogue of admitting a classifying space with finite n-skeleton. An essentially immediate consequence of Gaboriau's theorem (or [Lüc02, Theorem 7.2]) is that if G admits a virtual algebraic fibration with kernel K of type $\mathrm{FP}_n(\mathbb{Q})$, then $b_i^{(2)}(G) = 0$ for all $i \leqslant n$. For the class of RFRS groups, we have the following converse, which will be proved in Theorem 3.2.3.

Theorem 1.1.2. *Let G be a RFRS group of type* $\mathrm{FP}_n(\mathbb{Q})$*. There exists a finite-index subgroup $H \leqslant G$ and an epimorphism $H \to \mathbb{Z}$ with kernel of type* $\mathrm{FP}_n(\mathbb{Q})$ *if and only if $b_i^{(2)}(G) = 0$ for all $i \leqslant n$.*

Note that a group is of type $\mathrm{FP}_1(\mathbb{Q})$ if and only if it is finitely generated, so Theorem 1.1.2 reduces to Theorem 1.1.1 in the case $n = 1$. We now discuss some applications of Theorem 1.1.2. An obvious question raised by Thurston's Virtual Fibring Conjecture is whether higher-dimensional hyperbolic manifolds virtually fibre over S^1. In even dimensions this can never be the case: the Chern–Gauss–Bonnet Theorem implies that even-dimensional hyperbolic manifolds have non-zero Euler characteristic, but the Euler characteristic of any virtually fibred manifold is zero. Thus, the question is only interesting in odd dimensions, and until recently there was no evidence in favour or against a higher-dimensional analogue of the Virtual Fibring Conjecture being true. In [IMM23], Italiano, Martelli, and Migliorini constructed the first example of a finite-volume, cusped, hyperbolic 5-manifold that fibres over the circle, providing the first piece of evidence towards the conjecture. In the other direction, in [AOS24] Avramidi, Okun, and Schreve build a negatively curved (but not hyperbolic) closed 7-manifold that does not virtually fibre over S^1. Note that the manifolds in [IMM23] and [AOS24] have virtually RFRS fundamental groups.

As a consequence of Theorem 1.1.2, together with the fact that hyperbolic manifolds have their L^2-Betti numbers concentrated in the middle dimension [Dod79], we give the following evidence towards a higher-dimensional Virtual Fibring Conjecture.

Corollary 1.1.3. *Let M be an odd-dimensional hyperbolic manifold such that $\pi_1(M)$ is virtually RFRS. Then $\pi_1(M)$ virtually algebraically fibres with kernel of type* $\mathrm{FP}(\mathbb{Q})$.

In [BHW11], Bergeron, Haglund, and Wise show that lattices of simple type (see their paper for a definition) in $\mathrm{SO}(n,1)$ act geometrically on $\mathrm{CAT}(0)$ cube complexes, and therefore the uniform such lattices are virtually RFRS by [Ago13]. This provides a rich source of examples of manifolds satisfying the assumption of the previous corollary.

Corollary 1.1.4. *If n is odd and $\Gamma < \mathrm{SO}(n,1)$ is a uniform lattice of simple type, then Γ virtually algebraically fibres with kernel of type* $\mathrm{FP}(\mathbb{Q})$.

Remark 1.1.5. In [Kud23], Kudlinska shows that the type $\mathrm{FP}(\mathbb{Q})$ kernels appearing in the corollary above are not hyperbolic, giving examples of hyperbolic groups of all odd cohomological dimensions containing non-hyperbolic subgroups of type $\mathrm{FP}(\mathbb{Q})$. Note that an important consequence of [IMM23] is that there exists a hyperbolic group with a non-hyperbolic subgroup of finite type.

As remarked by Llosa-Isenrich, Martelli, and Py in [IMP24, Proposition 19], if Γ is a uniform lattice of simple type in $\mathrm{SO}(n,1)$ for n even, then Γ virtually algebraically fibres with kernel of type $\mathrm{FP}_{\frac{n}{2}-1}(\mathbb{Q})$ but not of type $\mathrm{FP}_{\frac{n}{2}}(\mathbb{Q})$, giving a homological solution to a question of Brady about the existence of such subgroups of hyperbolic groups [Bra99]. In fact, every lattice in $\mathrm{SO}(n,1)$ is of simple type when n is even (see [KPV08, Remark 2.1]), so we obtain the following corollary.

Corollary 1.1.6. *If n is even, then every uniform lattice in $\mathrm{SO}(n,1)$ contains a subgroup of type $\mathrm{FP}_{\frac{n}{2}-1}(\mathbb{Q})$ but not of type $\mathrm{FP}_{\frac{n}{2}}(\mathbb{Q})$.*

Brady's question was completely answered by Llosa-Isenrich and Py who gave examples of subgroups of hyperbolic groups that are of type F_{n-1} but not F_n for all $n \geqslant 1$ [LP24] (recall that a group is of type F_n if it admits a classifying space with finite n-skeleton), though it is still not known whether the subgroups appearing in Corollary 1.1.6 are of type $\mathrm{F}_{\frac{n}{2}-1}$.

1.1.1 Positive characteristic analogues of L^2-Betti numbers

There are different ways to define L^2-invariants over fields which are not subfields of $\mathbb{C}$. This is often done via Lück Approximation: in [Lüc94], Lück showed that if G is a residually finite group of type $\mathrm{FP}_{n+1}(\mathbb{Q})$, then

$$b_i^{(2)}(G) = \lim_{j\to\infty} \frac{b_i(G_j;\mathbb{Q})}{[G:G_j]}$$

for any residual chain of finite-index normal subgroups $G = G_0 \geqslant G_1 \geqslant \dots$ and for all $i \leqslant n$. By replacing $\mathbb{Q}$ with an arbitrary field k in the above limit, one would hope to obtain a new invariant $b_i^{(2)}(G;k)$, however the limit of the sequence $\frac{b_i(G_j;k)}{[G:G_j]}$ is not known to exist in general, nor whether the limit should depend on the chosen residual chain in the cases where it does exist (see [AOS21, BK17] for interesting cases where the answers to these questions are known). One way around this is to simply define the k-L^2-Betti numbers of a group G of type $\mathrm{FP}_{n+1}(k)$ by

$$b_i^{(2)}(G;k) = \inf_{H\leqslant_{\text{f.i.}} G} \frac{b_i(H;k)}{[G:H]}.$$

for $i \leqslant n$.

By a combination of results of Jaikin-Zapirain [JZ21] and [LLS11, Theorem 0.2], this definition agrees with a more algebraic definition of k-L^2-Betti numbers in the case of RFRS groups. Recall that if G is a RFRS group, then $\mathbb{Q}[G]$ embeds into a division ring $\mathcal{D}_{\mathbb{Q}[G]}$ and the L^2-Betti numbers of G are the $\mathcal{D}_{\mathbb{Q}[G]}$-dimensions of the homology modules $\mathrm{H}_i(G; \mathcal{D}_{\mathbb{Q}[G]})$. By [JZ21, Corollary 1.3], if G is RFRS then any group algebra $k[G]$ embeds into a division ring $\mathcal{D}_{k[G]}$ that shares many formal properties with $\mathcal{D}_{\mathbb{Q}[G]}$ (see Section 2.3). More precisely, $\mathcal{D}_{k[G]}$ is Hughes-free and universal as a division $k[G]$-ring. One then defines the k-L^2-Betti numbers of a RFRS group by

$$b_i^{(2)}(G; k) = \dim_{\mathcal{D}_{k[G]}} \mathrm{H}_i(G; \mathcal{D}_{k[G]}).$$

The ground field k plays no role in our arguments, and we obtain the following more general version of Theorem 1.1.2.

Theorem 1.1.7. *Let G be a RFRS group of type $\mathrm{FP}_n(k)$ for some field k. There exists a finite-index subgroup $H \leqslant G$ and an epimorphism $H \to \mathbb{Z}$ with kernel of type $\mathrm{FP}_n(k)$ if and only if $b_i^{(2)}(G; k) = 0$ for all $i \leqslant n$.*

1.2 Cohomological dimension of normal subgroups

Gaboriau's theorem [Gab02, Théorème 6.6] discussed in the previous section also implies that L^2-Betti numbers obstruct the existence of low-dimensional normal subgroups $N \trianglelefteqslant G$ such that G/N is amenable. More precisely, if $\mathrm{cd}_{\mathbb{Q}}(N) < n$, then $b_i^{(2)}(N) = 0$ for all $i \geqslant n$ and consequently $b_i^{(2)}(G) = 0$ for all $i \geqslant n$. In Chapter 4, we will give partial converses to this statement. We first state a special case of the result which we believe is of the most interest.

Theorem 1.2.1. *Let G be a finitely generated RFRS group of cohomological dimension at most two. Then G is virtually free-by-cyclic if and only if $b_2^{(2)}(G) = 0$.*

This generalises and gives a new proof of a theorem of Kielak and Linton [KL24], which has the additional assumptions that G be hyperbolic and virtually compact special. Note however, that Kielak and Linton prove that G is virtually a subgroup of a (finitely generated free)-by-cyclic group, while this does not follow from Theorem 1.2.1.

By a celebrated theorem of Feighn and Handel [FH99], free-by-cyclic groups are coherent, meaning that all their finitely generated subgroups are finitely presented. In [Bau74], Baumslag raised the problem as to whether all one-relator groups are coherent, and in [Bau86, Problem 6], he conjectured that in fact one-relator groups with torsion are virtually free-by-cyclic. More generally, Wise conjectures that hyperbolic one-relator groups are virtually free-by-cyclic [Wis20a, Conjecture 17.8]. There has been much recent progress in our understanding of all these conjectures. Jaikin-Zapirain and Linton [JZ17] proved that one-relator groups are coherent. Moreover, all one-relator groups have vanishing second L^2-Betti number by [DL07], and since one-relator groups with torsion are hyperbolic [New68] and virtually compact special [Wis21, Corollary 19.2], Kielak and Linton's result mentioned above implies that they are virtually free-by-cyclic. An immediate corollary of Theorem 1.2.1 is that any virtually RFRS one-relator group is virtually free-by-cyclic. It is likely the case that hyperbolic one-relator groups are virtually special, which would resolve Wise's conjecture, but this is not yet known to be the case.

In [Wis20b], Wise proves that if X is a compact two-complex with RFRS fundamental group, then $b_2^{(2)}(\widetilde{X}) \leqslant b_2(X)$. This was subsequently strengthened by Jaikin-Zapirain, who proved the same inequality for higher L^2-Betti numbers [JZ21, Corollary 1.6]; we will also prove a variant of this results in Lemma 4.3.13 which relaxes the assumption that X be compact. As a consequence of Theorem 1.2.1, we obtain the following corollary which solves a problem of Wise [Wis20b, Problem 6.5].

Corollary 1.2.2. *Let G be a finitely generated RFRS group of cohomological dimension at most two. If $b_2(G) = 0$, then G is virtually free-by-cyclic.*

Note that Hagen and Wise [HW10a] show that if G is a graph of free groups with cyclic edge groups or a limit group and G is hyperbolic relative to virtually $\mathbb{Z}^2$ subgroups, then G is virtually free-by-cyclic. More generally, they show that any special group with an elementary hierarchy is virtually free-by-cyclic (we refer the reader to their paper for a definition). Theorem 1.2.1 gives a new proof of these facts.

We now state a more general result that applies in all finite cohomological dimensions and to a class of groups larger than RFRS groups. Recall that the class of finitely generated RFRS groups coincides with the class of residually (poly-$\mathbb{Z}$ and virtually Abelian groups). We now drop the virtually Abelian assumption and consider the class of residually poly-$\mathbb{Z}$ groups. The following result was obtained in joint work with Sánchez-Peralta and generalises the previous work of the author with Klinge [FK24], where the assumption that G be residually (poly-$\mathbb{Z}$ and virtually nilpotent) was needed.

Theorem 1.2.3. *Let k be a field and let G be a residually poly-$\mathbb{Z}$ group of type* $\mathrm{FP}(k)$*. Let $n = \mathrm{cd}_k(G)$. The following are equivalent:*

1. *in every residual normal chain $G = G_0 \geqslant G_1 \geqslant \dots$ such that G/G_i is poly-$\mathbb{Z}$, $\mathrm{cd}_k(G_i) < n$ for all sufficiently large integers i;*
2. $b_n^{(2)}(G; k) = 0$.

Note that we do not need to pass to a finite-index subgroup in the conclusion of this theorem, as opposed to in Theorem 1.2.1. The finiteness assumption can be relaxed in two different ways. The arguments actually show that the conclusion of Theorem 1.2.3 holds when the trivial $k[G]$-module k admits a projective resolution of length n such that the top-dimensional module is finitely generated. It also follows from the arguments of Jaikin-Zapirain and Linton [JZL23] that if G satisfies $b_n^{(2)}(G;k) = 0$, is of type $\mathrm{FP}_{n-1}(k)$, and $\mathrm{cd}_k(G) = n$, then in fact G is of type $\mathrm{FP}(k)$; this is the reason why we only require G to be finitely generated in Theorem 1.2.1, otherwise the assumption that G be of type $\mathrm{FP}_2(\mathbb{Q})$ would be needed.

Returning to two-dimensional groups, the theorem of Stallings and Swan [Sta68, Swa69], together with Theorem 1.2.3, implies the following corollary. We say that a ring is *coherent* if all of its one-sided ideals are finitely presented. While this clashes with usual terminology, we deem it appropriate here since group algebras are left coherent (meaning all their finitely generated left ideals are finitely presented) if and only if they are right coherent.

Corollary 1.2.4. *Let G be a finitely generated residually poly-$\mathbb{Z}$ group of cohomological dimension at most two. Then G is free-by-(poly-$\mathbb{Z}$) if and only if $b_2^{(2)}(G) = 0$. In particular, if G is residually poly-$\mathbb{Z}$, of cohomological dimension at most 2, and $b_2^{(2)}(G) = 0$, then G is coherent and the group algebra $k[G]$ is coherent for any field k.*

The fact that free-by-(poly-$\mathbb{Z}$) groups of cohomological dimension two are coherent and have coherent group algebras follows easily from the arguments in [JZL23] (see Corollary 4.3.11 for a proof).

Corollary 1.2.4 fits into a conjectural classification of two dimensional coherent groups, proposed by Gromov and Wise (see [Wis20a, Wis22a, Wis22b]). A map $Y \to X$ between 2-complexes X and Y is a *combinatorial immersion* if it is cellular and is injective on the link of each vertex. A 2-complex X has *non-positive immersions* if for every combinatorial immersion $Y \to X$, either Y is contractible or $\chi(Y) \leqslant 0$.

Conjecture 1.2.5. *Let G be a group of geometric dimension at most two. The following are equivalent:*

(1) *G is coherent;*
(2) *$b_2^{(2)}(G) = 0$;*
(3) *$G \cong \pi_1(X)$ for a 2-complex X with non-positive immersions.*

Corollary 1.2.4 shows that item (2) implies item (1) in the class of residually poly-$\mathbb{Z}$ groups. Theorem 1.2.1 shows that the implication (2) $\Rightarrow$ (3) holds virtually in the class of RFRS groups, i.e. if $b_2^{(2)}(G) = 0$ for a two-dimensional RFRS group G, then there is a finite-index subgroup $H \leqslant G$ such that H is the fundamental group of a 2-complex with non-positive immersions. This is because the standard presentation complex of a free-by-cyclic group (more generally, of an ascending HNN extension of a free group) has non-positive immersions by [Wis22a, Theorem 6.1]. While no implication is known in general, Jaikin-Zapirain and Linton [JZL23] show that if G is a group of cohomological dimension at most two satisfying the Strong Atiyah Conjecture and $b_2^{(2)}(G) = 0$, then G is *homologically coherent*, meaning that the finitely generated subgroups of G are of type $\mathrm{FP}_2(\mathbb{Z})$. They are also able to promote the homological coherence property to full coherence in many cases of interest, such as that of one-relator groups (see [JZL23, Section 4]). There is much less evidence for either of the implications (1) $\Rightarrow$ (2) or (1) $\Rightarrow$ (3), and it seems less likely that these would hold in general (this is also Wise's view). In fact, the group $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ is a potential counterexample; whether it is coherent is a well-known open problem of Serre's, while it is known that it has non-vanishing second L^2-Betti number.

1.2.1 Parafree groups

We conclude this section with a short discussion on parafree groups and the Parafree Conjecture. A group G is *parafree* if G is residually nilpotent and there exists a free group F such that $G/\gamma_n(G) \cong F/\gamma_n(F)$ for all n, where $\gamma_n(G)$ denotes thee nth term of the lower central series of G. There are many examples of finitely generated non-free parafree groups G [Bau67]. The main open problem in the subject is Baumslag's Parafree Conjecture, which predicts that $\mathrm{H}_2(G;\mathbb{Z}) = 0$ for every finitely generated parafree group G. The additional prediction that $\mathrm{H}_2(G;\mathbb{Z}) = 0$ and $\mathrm{cd}(G) \leqslant 2$ is sometimes called the Strong Parafree Conjecture; both versions are open. Another important question about finitely generated parafree groups is whether they are finitely presented, and it is even unknown whether they are coherent. Note that it is known that finite presentability is not a pronilpotent invariant by a result of Bridson and Reid [BR15, Theorem A]. Using Corollary 1.2.4, we draw a relation between these questions.

Corollary 1.2.6. *Let G be a finitely generated parafree group with* $\mathrm{cd}(G) \leqslant 2$*. The following are equivalent:*

1. *G satisfies the Parafree Conjecture;*
2. *G is free-by-(free nilpotent);*
3. *the terms of the lower central series of G are eventually free.*

Hence, if G satisfies the Parafree Conjecture, then G is coherent, and in particular finitely presented.

Recall that a free nilpotent group is any quotient of a free group F by some term $\gamma_n(F)$ of its lower central series.

1.3 Constructing division rings

The results discussed above all rely on the existence of embeddings of the group algebra into a suitable division ring, and it is thus of interest to extend the list of groups whose group algebras admit embeddings into division rings. Note that if $k[G]$ embeds into a division ring, then $k[G]$ has no zero divisors, so embeddability into a division ring implies Kaplansky's famous Zero Divisor Conjecture.

Conjecture 1.3.1 (The Zero Divisor Conjecture)**.** *Let G be a torsion-free group and let k be a field. Then $k[G]$ has no zero divisors.*

The assumption that G is torsion-free is easily seen to be necessary, and therefore it will also be necessary for $k[G]$ to embed into a division ring. Somewhat surprisingly, the following a priori stronger conjecture remains open.

Conjecture 1.3.2. *Let G be a torsion-free group and let k be a field. Then $k[G]$ embeds into a division ring.*

An even stronger version of this was conjectured in [JZL23, Conjecture 1]. Conjecture 1.3.2 is related to *Malcev's Problem*, which asks whether group algebras of left-orderable groups embed into division rings. This was motivated by Malcev and Neumann's independent constructions of division rings containing the group algebras of bi-orderable groups [Mal48, Neu49]. Other notable classes of groups satisfying the conclusion of Conjecture 1.3.2 include torsion-free elementary amenable groups [KLM88] and locally indicable groups (when k is of characteristic zero) [JZLÁ20].

In the final chapter of this thesis, we extend the list of groups with group algebras embedding into division rings. This result was obtained jointly with Sánchez-Peralta in [FSP23].

Theorem 1.3.3. *If M is a 3-manifold with torsion-free fundamental group and k is a field, then $k[\pi_1(M)]$ embeds into a division ring.*

As a corollary, we deduce the Zero Divisor Conjecture in the class of 3-manifold groups.

Corollary 1.3.4. *If M is a 3-manifold with torsion-free fundamental group and k is a field, then $k[\pi_1(M)]$ has no zero divisors.*

We remark that Theorem 1.3.3 and Corollary 1.3.4 hold when the group algebra is replaced by any twisted group algebra $k * \pi_1(M)$ for any division ring k. When $k = \mathbb{C}$, both results follow from the resolution of the Strong Atiyah Conjecture for 3-manifold groups [FL19, KL24]. The Zero Divisor Conjecture for the rational group ring $\mathbb{Q}[G]$ of a 3-manifold group G was raised in [AFW15, Question 7.2.6(6)].

The proof of these results uses a mix of classical 3-dimensional topology, including the Prime and JSJ Decomposition Theorems, recent advances in the study of 3-manifolds, such as the resolution of the Virtual Fibring Conjecture, and the theory of coproducts and HNN extensions of rings developed by Cohn, Bergmann, and Dicks [Ber74, Coh06, Dic83]. We take special care to state the classical results in such a way that we do not need to assume that the 3-manifold M is orientable. Indeed, it is not sufficient to prove the results in the orientable case, since it is not known whether the Zero Divisor Conjecture nor Conjecture 1.3.2 are stable under passage to an overgroup of index two.

Along the way, we will prove that group algebras of torsion-free virtually compact special groups admit Linnell embeddings into division rings (see Definition 2.3.7 and Theorem 5.2.4), building on arguments of [LS07] and [Sch14]. This confirms a conjecture of Kielak and Linton [KL24, Conjecture 6.8]. The existence of Linnell embeddings of virtually compact special groups is a crucial step in establishing Theorem 1.3.3.

1.4 Organisation of the thesis

In Chapter 2, we introduce the preliminary material that will be used throughout the thesis. Many of the concepts that were not fully defined in the introduction are covered in more detail there. In Chapter 3, we prove Theorems 1.1.2 and 1.1.7 and discuss some applications of these results. In Chapter 4, we begin by giving a very short proof of Theorem 1.2.1, and then proceed to prove Theorem 1.2.3. Finally, in Chapter 5, we prove Theorem 1.3.3. The key tool in this section is the graph of rings construction, which is developed in Section 5.1. Two appendices are included at the end of the thesis. In Appendix A, we compute the k-L^2-Betti numbers of various locally indicable groups admitting embeddings into division rings. Appendix B contains a list of questions and conjectures related to the thesis.

Chapter 2

Preliminaries

2.1 Twisted group rings

Given a ring R and a group G, we can always form the group ring $R[G]$, whose underlying set consists of finite formal sums of elements of G with coefficients in R. We will also need the more general notion of a twisted group ring, which we now define. A ring S is G-*graded* if its underlying additive group decomposes as $S = \bigoplus_{g \in G} S_g$ and $S_g S_h \subseteq S_{gh}$ for all $g, h \in G$. If, additionally, there are distinguished units $s_g \in S_g$ for each $g \in G$, then we say that S is a *twisted group ring* of S_e and G, and denote it by $S_e * G$. By abuse of notation, we will often denote the distinguished unit $s_g \in S_g$ by g. Thus, elements of a twisted group ring $S_e * G$ will be written as finitely supported formal sums $\sum_{g \in G} \lambda_g g$, where $\lambda_g \in S_e$. Twisted group rings arise naturally and frequently, as the following example shows.

Example 2.1. Let R be a ring and G be a group. If $N \trianglelefteq G$ is a normal subgroup, then $R[G]$ is isomorphic to a twisted group ring $R[N] * G/N$.

2.2 Group homology and finiteness conditions

All of the material presented in this section is standard and can be found in [Bie81] or [Bro94].

Throughout the section, R denotes a unital, associative ring with $1 \neq 0$. A *resolution* of a left R-module M is an exact sequence

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where each P_n is an R-module and the maps are module homomorphisms. We can similarly define resolutions of right R-modules. For conciseness, we will use the notation $P_\bullet \to M$ to denote a resolution of this form. If $\mathcal{P}$ is a property of modules, then $P_\bullet \to M$ is a *$\mathcal{P}$-resolution* if all the modules P_n have the property $\mathcal{P}$.

Definition 2.2.1 (Module finiteness properties)**.** The R-module M is of *type* FP_n if there is a projective resolution $P_\bullet \to M$ such that P_i is finitely generated for all $i \leqslant n$. If there exists such a projective resolution such that all the modules P_i are finitely generated, then M is of *type* FP_∞. If, moreover, the resolution can be chosen so that only finitely many of the modules P_i are nonzero, then M is of type FP.

If G is a group and R is a ring, then R can viewed as an $R[G]$-module by making G act trivially. We call R the *trivial* $R[G]$-module.

Definition 2.2.2 (Group finiteness properties)**.** The group G is of *type* $\mathrm{FP}_n(R)$ (resp. *type* $\mathrm{FP}_\infty(R)$, resp. *type* $\mathrm{FP}(R)$) if the trivial $R[G]$-module R is of type FP_n (resp. FP_∞, resp. FP).

A group is of type $\mathrm{FP}_1(R)$ for some (and hence every) ring R if and only if it is finitely generated [Bie81, Proposition 2.1]. If G has a classifying space with finite n-skeleton, then it is of type $\mathrm{FP}_n(R)$ for all rings R, and similarly for the other finiteness conditions. If S is an R-algebra, then type $\mathrm{FP}_n(R)$ implies type $\mathrm{FP}_n(S)$ for all n, and similarly for the other finiteness properties. Since every ring is a $\mathbb{Z}$-algebra, this implies that $\mathrm{FP}_n(\mathbb{Z})$ implies $\mathrm{FP}_n(R)$ for all rings R. We say that G is of *finite type* if G admits a finite classifying space.

Definition 2.2.3 (The functors Tor and Ext)**.** Let M and N be right and left R-modules, respectively. The functors $\mathrm{Tor}_n^R(-, N)$ are the derived functors of $- \otimes_R N$. This means that $\mathrm{Tor}_n^R(M, N)$ is calculated by taking a projective resolution $P_\bullet \to M$ and computing the degree n homology of the chain complex $P_\bullet \otimes_R N$; the result is independent of the chosen resolution.

Now suppose that M and N are both left R-modules. The functors $\mathrm{Ext}_R^n(-, N)$ are the derived functors of $\mathrm{Hom}_R(-, N)$. This mean that $\mathrm{Ext}_n^R(M, N)$ is calculated by taking a projective resolution $P_\bullet \to M$ and computing the degree n cohomology of the cochain complex $\mathrm{Hom}_R(P_\bullet, N)$. Again, the result is independent of the chosen resolution.

Note that $\mathrm{Tor}_n^R(M, N)$ can also be calculated as the derived functor of $M \otimes_R -$, a fact which we will occasionally use without mention.

Definition 2.2.4 (Group (co)homology)**.** Let G be a group, R be a ring, and M be a left $R[G]$-module. The degree n *homology* of G with coefficients in M is

$$\mathrm{H}_n(G; M) \ := \ \mathrm{Tor}_n^{R[G]}(R, M).$$

The degree n *cohomology* of G with coefficients in M is

$$\mathrm{H}^n(G; M) \ := \ \mathrm{Ext}^n_{R[G]}(R, M).$$

The *length* of a projective resolution $P_\bullet \to M$ is $\sup\{n : P_n \neq 0\}$.

Definition 2.2.5 (Cohomological dimension)**.** The *cohomological dimension* of a group G over a ring R is the minimal length of a projective resolution of the trivial $R[G]$-module R. It is denoted by $\mathrm{cd}_R(G)$. If all resolutions of R are of infinite length, then $\mathrm{cd}_R(G) = \infty$. Equivalently, $\mathrm{cd}_R(G)$ is the maximal n such that there exists an R-module M with $\mathrm{H}^n(G; M) \neq 0$.

2.3 Division rings

Throughout the text, we will make use of various constructions of division rings, which we overview here. All rings are assumed to be unital and associative with $1 \neq 0$ and ring homomorphisms preserve the unit.

2.3.1 Ore localisation

Definition 2.3.1 (Ore domains)**.** Let R be a ring and let $S = R \smallsetminus \{0\}$. If R is a domain, then R is a *right Ore domain* if $sR \cap rS \neq \varnothing$ for all pairs $(r, s) \in R \times S$.

Given a right Ore domain R, we can form its *Ore localisation* $\mathrm{Ore}(R)$ as follows. Continuing with the notation $S = R \smallsetminus \{0\}$, define an equivalence relation on $R \times S$ by $(r, s) \sim (r', s')$ if and only if there are elements $\sigma, \sigma' \in S$ such that $r\sigma = r'\sigma'$ and $s\sigma = s'\sigma'$. We denote the equivalence class of (r, s) under $\sim$ by r/s, and we call the equivalence classes *right fractions*. Let r_1/s_1 and r_2/s_2 be right fractions. Since R is a right Ore domain, there are elements $\sigma, \sigma' \in S$ such that $s_1\sigma = s_2\sigma'$. We can then define the sum of r_1/s_1 and r_2/s_2 by

$$r_1/s_1 + r_2/s_2 \ := \ (r_1\sigma + r_2\sigma')/s_1\sigma.$$

Similarly, there are elements $\tau, \tau' \in S$ such that $s_1\tau = r_2\tau'$, which we can use to define the multiplication of r_1/s_1 and r_2/s_2 by

$$(r_1/s_1) \cdot (r_2/s_2) \ := \ (r_1\tau)/(s_2\tau').$$

It is an arduous and tedious task to check that the definitions of addition and multiplication do not depend on the right fraction representatives and that they define a ring structure on $\mathrm{Ore}(R)$. Even once one has established that the definitions are well defined, it is not trivial to show that addition is commutative. Once this is done however, it is easy to see the following.

Proposition 2.3.2. *Let R be a right Ore domain. Then $\mathrm{Ore}(R)$ is a division ring and the map $\iota\colon R \to \mathrm{Ore}(R)$, $r \mapsto r/1$ is a ring monomorphism.*

Ore localisation also satisfies the following universal property, which is analogous to the universal property of localisation of commutative rings. We use $D^\times$ to denote the set of units in a ring D.

Proposition 2.3.3. *Let R be an Ore domain and let $f\colon R \to D$ be a ring homomorphism satisfying $f(S) \subseteq D^\times$. There is a unique extension $\overline{f}\colon \mathrm{Ore}(R) \to D$ satisfying $f = \overline{f} \circ \iota$.*

A *left Ore domain* is a domain R where $Rs \cap Sr \neq \varnothing$ for all $(r, s) \in R \times S$ (again, $S = R \smallsetminus \{0\}$). Since we will only be interested in the case where R is a (twisted) group ring, and in this case R is a left Ore domain if and only if it is a right Ore domain, this distinction is not important for us. The universal property ensures that the left and right Ore localisations are isomorphic rings. For more details on Ore domains and the more general Ore condition, we refer the reader to [Pas77, Section 4.4]. We will make frequent use (often without mention) of the following result.

Theorem 2.3.4 ([KLM88, Theorem 1.4])**.** *Let k be a division ring and let G be a torsion-free elementary amenable group. Then all twisted group rings $k * G$ are Ore domains.*

2.3.2 Malcev–Neumann power series

A group G is *orderable* if there is a total order $<$ on G such that $g_1 < g_2$ if and only if $tg_1 < tg_2$ if and only if $g_1 t < g_2 t$ for all elements $g_1, g_2, t \in G$. If G is orderable and $<$ is a fixed order on G, we refer to the pair $(G, <)$ as an ordered group (often we will just say that G is ordered when $<$ is implicit).

Definition 2.3.5 (The Malcev–Neumann power series ring)**.** Let $R * G$ be a twisted group ring of an ordered group $(G, <)$ and a ring R. The *Malcev–Neumann power series ring* associated to $R * G$ and $<$ is the ring with underlying set

$$R *_< G \;=\; \left\{ x = \sum_{g \in G} r_g g \;:\; r_g \in R \text{ and } \operatorname{supp}(x) \text{ is well ordered} \right\}.$$

Addition and multiplication in $R * G$ are well defined and naturally extend the corresponding operations in $R * G$.

The following theorem of Malcev and Neumann gives many examples of group rings embedding into division rings.

Theorem 2.3.6 ([Mal48, Neu49]). *Let $k * G$ be a twisted group ring of an ordered group G and division ring k. Then $k *_< G$ is a division ring.*

2.3.3 Linnell and Hughes-free division rings

In this subsection, we fix the following notation. Let $k * G$ be a twisted group ring of a group G and a division ring k, and suppose there is an embedding $\iota\colon k * G \hookrightarrow \mathcal{D}$, where $\mathcal{D}$ is a division ring. Note that ι gives $\mathcal{D}$ the structure of a $k * G$ bi-module. If $H \leqslant G$ is a subgroup, we denote the division closure of $\iota(k * H)$ in $\mathcal{D}$ by $\mathcal{D}_H$.

Definition 2.3.7 (Linnell division rings). The embedding ι is *Linnell* if $\mathcal{D} = \mathcal{D}_G$ and the multiplication map

$$\mathcal{D}_H \otimes_{k*H} (k * G) \to \mathcal{D}, \quad \alpha \otimes x \mapsto \alpha \cdot \iota(x)$$

is injective for all subgroups $H \leqslant G$. In this situation, we say that $\mathcal{D}$ is a *Linnell division ring* for $k * G$.

If ι is Linnell, then it follows that the restriction $k * H \to \mathcal{D}_H$ is Linnell for every subgroup $H \leqslant G$. There is also the following, a priori weaker, type of embedding that is very useful. Recall that a group G is *locally indicable* if every non-trivial finitely generated subgroup $H \leqslant G$ admits an epimorphism to $\mathbb{Z}$.

Definition 2.3.8. If G is locally indicable, then the embedding ι above is *Hughes-free* if $\mathcal{D} = \mathcal{D}_G$ and the multiplication map

$$\mathcal{D}_N \otimes_{k*N} (k * H) \to \mathcal{D}$$

is injective for all pairs of subgroups $N \trianglelefteqslant H$ of G such that H is finitely generated and $H/N \cong \mathbb{Z}$. In this situation, we say that $\mathcal{D}$ is a *Hughes-free division ring* for $k * G$.

Hughes-free embeddings are particularly useful because of the following result of Hughes.

Theorem 2.3.9 ([Hug70]). *If G is locally indicable and $k * G$ has a Hughes-free division ring, then it is unique up to isomorphism of $k * G$-algebras.*

In view of this result, if a Hughes-free embedding exists, then we denote it by $\mathcal{D}_{k*G}$. Then $k * H$ has a Hughes-free embedding for every subgroup $H \leqslant G$ and $\mathcal{D}_{k*H} \subseteq \mathcal{D}_{k*G}$. The following recent result of Gräter will be useful.

Theorem 2.3.10 ([Grä20, Corollary 8.3]). *Hughes-free division rings are Linnell.*

An immediate consequence of this result and Theorem 2.3.4 is the following.

Corollary 2.3.11. *Suppose that G is locally indicable and $\mathcal{D}_{k*G}$ exists.*

(i) *If $N \trianglelefteqslant G$ is a normal subgroup such that G/N is torsion-free and elementary amenable, then $\mathcal{D}_{k*G} \cong \mathrm{Ore}(\mathcal{D}_{k*N} * G/N)$.*

(ii) *If $N \trianglelefteqslant G$ is a normal subgroup of finite-index, then $\mathcal{D}_{k*G} \cong \mathcal{D}_{k*N} * G/N$.*

Implicit in the above statement is the fact that the twisted group ring structure $(k*N)*G/N$ extends to $\mathcal{D}_{k*N}*G/N$, which follows from the uniqueness of Hughes-free division rings (see [Hug70, p. 183]). Note that it is not known whether Linnell division rings are unique when they exist, which is the reason why Hughes-free division rings of locally indicable groups will play a more central role.

2.3.4 The category of R-division rings and specialisations

Let R be a ring. An *R-division ring* is a homomorphism $\varphi\colon R \to \mathcal{D}$, where $\mathcal{D}$ is a division ring and the image $\iota(R)$ generates $\mathcal{D}$ as a division ring. We will often, by abuse of notation, use $\mathcal{D}$ to denote the R-division ring $\varphi\colon R \to \mathcal{D}$.

Definition 2.3.12 (Specialisations). A *specialisation* from an R-division ring $\mathcal{D}_1$ to an R-division ring $\mathcal{D}_2$ is a homomorphism $\rho\colon D \to \mathcal{D}_2$, where $D \subseteq \mathcal{D}_1$ is a local subring of $\mathcal{D}_1$ containing the image of R, the maximal ideal of D is $\ker \rho$, and the diagram

$$\begin{array}{ccccc} & & R & & \\ & \swarrow & \downarrow & \searrow & \\ \mathcal{D}_1 & \hookleftarrow & D & \xrightarrow{\rho} & \mathcal{D}_2 \end{array}$$

commutes.

We will denote the specialisation by $\rho\colon \mathcal{D}_1 \to \mathcal{D}_2$, even though ρ is not a map with domain $\mathcal{D}_1$ in general. If there are specialisations $\rho_1\colon \mathcal{D}_1 \to \mathcal{D}_2$ and $\rho_2\colon \mathcal{D}_2 \to \mathcal{D}_1$, then we may take the domain of ρ_i to be all of $\mathcal{D}_i$ $(i = 1, 2)$, which forces the maps ρ_i

to be inverse to each other. Hence, we may define a partial order on the isomorphism classes of R-division rings by declaring $\mathcal{D}_1 \geqslant \mathcal{D}_2$ if and only if there is a specialisation $\mathcal{D}_1 \to \mathcal{D}_2$.

There is another partial order on the isomorphism classes of R-division rings that we can define as follows. Given an R-division ring $\varphi\colon R \to \mathcal{D}$ and a matrix M with coefficients in R, applying φ to the entries of M yields a matrix M^φ with coefficients in $\mathcal{D}$. Define the *φ-rank* of M to be the number of $\mathcal{D}$-linearly independent columns in M. We denote this quantity by $\operatorname{rk}_{\mathcal{D}}(M)$, since the map φ will always be understood. In this way, an R-division ring $\mathcal{D}$ determines a function $\operatorname{rk}_{\mathcal{D}}\colon \operatorname{Mat}(R) \to \mathbb{Z}_{\geqslant 0}$, where $\operatorname{Mat}(R)$ denotes the set of all finite matrices over R, and we obtain a new poset on the isomorphism classes of R-division rings by declaring $\mathcal{D}_1 \geqslant' \mathcal{D}_2$ if and only if $\operatorname{rk}_{\mathcal{D}_1}(M) \geqslant \operatorname{rk}_{\mathcal{D}_2}(M)$ for all matrices M with coefficients over R. A theorem of Malcolmson states that the two posets we have introduced are isomorphic.

Theorem 2.3.13 ([Mal80, Theorem 2])**.** *If $\mathcal{D}_1$, $\mathcal{D}_2$ are R-division rings, then $\mathcal{D}_1 \geqslant \mathcal{D}_2$ if and only if $\mathcal{D}_1 \geqslant' \mathcal{D}_2$. In other words, there is a specialisation $\mathcal{D}_1 \to \mathcal{D}_2$ if and only if the induced rank functions satisfy $\operatorname{rk}_{\mathcal{D}_1} \geqslant \operatorname{rk}_{\mathcal{D}_2}$.*

An R-division ring $\mathcal{D}$ is *universal* if $\mathcal{D} \geqslant \mathcal{D}'$ for all other R-division rings $\mathcal{D}'$. The following result of Jaikin-Zapirain gives many examples of universal $k[G]$-division rings, and will be used throughout the thesis.

Theorem 2.3.14 ([JZ21, Corollary 1.3])**.** *Let G be a residually (locally indicable and amenable) group and let k be a division ring. Then the Hughes-free division ring $\mathcal{D}_{k[G]}$ exists and is universal.*

2.4 RFRS groups

Definition 2.4.1 (RFRS)**.** A group G is *residually finite rationally solvable (RFRS)* if there is a residual chain of finite-index subgroups $G = G_0 \geqslant G_1 \geqslant G_2 \geqslant \dots$ such that $\ker(G_i \to \mathbb{Q} \otimes G_i/[G_i, G_i]) \leqslant G_{i+1}$ for all $i \geqslant 0$. We refer to a chain of this type as a *witnessing chain.*

The class of RFRS groups was introduced by Agol in connection with Thurston's Virtual Fibring Conjecture for 3-manifolds [Ago08], where he showed that a finite-volume hyperbolic 3-manifold has a finite degree cover that fibres over S^1 if $\pi_1(M)$ is virtually RFRS. Since then, RFRS groups have occupied a central role in geometric group theory, and there are now many examples of RFRS groups. For example, right-angled Artin groups are known to be RFRS, and it is clear that the RFRS property passes to subgroups. Hence, all special groups (in the sense of Haglund–Wise [HW08]) are RFRS.

Recently, Kielak, Okun, Schreve, and the author obtained a characterisation of the RFRS property in terms of an easy-to-parse residual property [OS24, Theorem 6.3]. We repeat the argument here. Recall that a group G is *poly-$\mathbb{Z}$* if there is a subnormal series $\{1\} = G_n \leqslant \cdots \leqslant G_1 \leqslant G_0 = G$ such that $G_i/G_{i+1} \cong \mathbb{Z}$ for each relevant index i.

Theorem 2.4.2. *A finitely generated group G is RFRS if and only if G is residually (poly-$\mathbb{Z}$ and virtually Abelian).*

Proof. Assume first that G is RFRS and let $G = G_0 \geqslant G_1 \geqslant \dots$ be a witnessing chain. Moreover, let $N_0 = G$ and $N_{i+1} = \ker(G_i \to \mathrm{H}_1(G_i;\mathbb{Q}))$ for all $i \geqslant 1$. Then $N_i/N_{i+1} \leqslant G_i/N_{i+1}$ is free Abelian of finite rank, which implies that G/N_{i+1} is poly-$\mathbb{Z}$ for all i. Moreover, G_i/N_{i+1} is free Abelian, and therefore G/N_{i+1} is poly-$\mathbb{Z}$ virtually Abelian.

Conversely, assume that G is residually (poly-$\mathbb{Z}$ virtually Abelian). The result will follow quickly from the two following claims.

Claim 2.4.3. *If G is poly-$\mathbb{Z}$ virtually Abelian, then G is RFRS.*

Proof. We prove the claim by induction on the poly-$\mathbb{Z}$ length of G. The base case is trivial, so we assume there are $n > 1$ cyclic factors. Then G fits into an extension $1 \to P \to G \to \mathbb{Z} \to 1$, where P is RFRS by induction. Let $P = P_0 \geqslant P_1 \geqslant \dots$ be a witnessing chain for the RFRS property, and assume that each subgroup P_i is characteristic, which can be done by passing to normal cores. Since G is virtually Abelian, there is an integer $n_1 > 1$ such that $n_1\mathbb{Z} \leqslant \mathbb{Z}$ acts trivially on the free Abelianisation P^{fab}. Let $G_1 = P \rtimes n_1\mathbb{Z}$, and note that $\ker(G \to G^{\mathrm{fab}}) \leqslant G_1$.

There is some $n_2 > n_1$ such that $n_2\mathbb{Z} \leqslant n_1\mathbb{Z}$ acts trivially on P_1^{fab}, and we set $G_2 = P_1 \rtimes n_2\mathbb{Z}$. Note that

$$\ker(G_1 \to G_1^{\mathrm{fab}}) = \ker(P \to P^{\mathrm{fab}}) \leqslant P_1 \leqslant G_2$$

since $(P_i)_{i\geqslant 0}$ is a witnessing chain. Continuing in this way, we obtain a chain of finite-index subgroups $G \geqslant G_1 \geqslant G_2 \geqslant \dots$, where $G_i = P_{i-1} \rtimes n_i\mathbb{Z}$ for some strictly increasing sequence of integers n_i such that $n_i\mathbb{Z}$ acts trivially on P_{i-1}^{fab}. The chain satisfies the condition $\ker(G_i \to G_i^{\mathrm{fab}}) \leqslant G_{i+1}$, and it is clearly residual. Thus, G is RFRS. ◇

Claim 2.4.4. *If G_n is a RFRS group for every $n \in \mathbb{N}$, then $\prod_n G_n$ is RFRS.*

Proof. For each $n \in \mathbb{N}$, let $G_n = G_n^{(0)} \geqslant G_n^{(1)} \geqslant \dots$ be a residual chain witnessing the RFRS property. Then the groups

$$G^{(n)} = G_1^{(n)} \times G_2^{(n-1)} \times \cdots \times G_{n-1}^{(2)} \times G_n^{(1)} \times G_{n+1} \times G_{n+2} \times \cdots$$

are of finite index in $\prod_n G_n$, and $G^{(1)} \geqslant G^{(2)} \geqslant \dots$ is a witnessing chain. $\diamond$

Since G is finitely generated and residually (poly-$\mathbb{Z}$ virtually Abelian), it follows that G embeds into a countable product $\prod_{i \in \mathbb{N}} G_i$ of poly-$\mathbb{Z}$ virtually Abelian groups G_i, which is RFRS by the two claims above. Since the RFRS property passes to subgroups, G is RFRS. $\square$

It is well known that nilpotence and the RFRS condition are incompatible, in the sense that the only nilpotent RFRS groups are Abelian. We give an easy proof of the following more general fact. A different proof was originally communicated to the author by Sami Douba, and appears in [Fis24a, Theorem 7.4].

Corollary 2.4.5. *If G is polycyclic and RFRS, then G is virtually Abelian. In particular, if G is nilpotent and RFRS, then G is Abelian.*

Proof. By passing to a finite-index subgroup, we may assume that G is poly-$\mathbb{Z}$. Let $G = N_0 > N_1 > \dots$ be a residual normal chain such that G/N_i is non-trivial poly-$\mathbb{Z}$ virtually Abelian for each i. Then $\mathrm{cd}_{\mathbb{Z}}(N_{i+1}) < \mathrm{cd}_{\mathbb{Z}}(N_i)$ for each i and thus N_i is trivial for sufficiently large i. Hence, G is virtually Abelian.

Now suppose G is nilpotent and RFRS. It suffices to prove the claim when G is finitely generated, in which case G is polycyclic. By the previous paragraph, G is virtually Abelian. But a virtually Abelian torsion-free nilpotent group is Abelian. $\square$

Note that $\mathbb{Z} \wr \mathbb{Z}$ is an example of a solvable RFRS group, so the corollary does not extend to the class of solvable groups. We will prove, however, that amenable RFRS groups of finite type must be virtually Abelian (see Corollary 3.2.4).

2.5 L^2-Betti numbers

In this section, we give the algebraic perspective on L^2-Betti numbers that will be used throughout the thesis. One advantage of this point of view is that it is independent of the characteristic of the ground field, while the classical L^2-theory requires that one work over a subfield of $\mathbb{C}$. The downside is a loss of generality, since L^2-Betti numbers can be defined for any group, while the version of the theory we are using only works for groups G such that $k[G]$ admits a Hughes-free embedding. For a treatment of the classical theory of L^2-invariants, the reader is referred to [Lüc02] and [Kam19].

Let G be a locally indicable group and k a field such that there is a Linnell embedding $k[G] \hookrightarrow \mathcal{D}_{k[G]}$.

Definition 2.5.1 (k-L^2-Betti numbers)**.** The *k-L^2-homology* of G in degree n is

$$\mathrm{H}_n^{(2)}(G; k) \ := \ \mathrm{H}_n(G; \mathcal{D}_{k[G]}).$$

Then $\mathrm{H}_n^{(2)}(G; k)$ is a $\mathcal{D}_{k[G]}$-module, and therefore it has a well-defined dimension. We thus define the nth *k-L^2-Betti number* of G by $b_n^{(2)}(G; k) := \dim_{\mathcal{D}_{k[G]}} \mathrm{H}_n^{(2)}(G; k)$.

The k-L^2-cohomology of G will also play a role. It is defined by

$$\mathrm{H}^n_{(2)}(G; k) := \mathrm{H}^n(G; \mathcal{D}_{k[G]})$$

and we define the cohomological k-L^2-Betti numbers $b^n_{(2)}(G; k) := \dim_{\mathcal{D}_{k[G]}} \mathrm{H}^n_{(2)}(G; k)$.

2.5.1 The Atiyah Conjecture

We now justify the terminology of the previous section by recalling some facts about the usual L^2-invariants of a group. Any group G acts by left multiplication on $L^2(G)$, and therefore $\mathbb{C}[G]$ is naturally a sub-algebra of the algebra of bounded operators on $L^2(G)$. The set of bounded operators on $L^2(G)$ that commute with the G-action is denoted by $\mathcal{N}(G)$ and called the von Neumann algebra of G. The set of non-zero divisors in $\mathcal{N}(G)$ is a left and right Ore set by [Ber82], and thus we can form the Ore localisation $\mathcal{U}(G)$, which is called the algebra of unbounded operators affiliated to G. It is possible to define a rank function for modules over $\mathcal{U}(G)$, and the strong Atiyah Conjecture over $\mathbb{C}$ is a prediction about the possible values of the rank of a matrix with entries in $\mathbb{C}[G]$. We are interested in the following equivalent formulation of the conjecture due to Linnell [Lin93].

Definition 2.5.2 (The Atiyah Conjecture)**.** A torsion-free group G satisfies the *Strong Atiyah Conjecture over* $\mathbb{C}$ if and only if the division closure of $\mathbb{C}[G]$ in $\mathcal{U}(G)$ is a division ring.

In general, the division closure of $\mathbb{C}[G]$ in $\mathcal{U}(G)$ is called the *Linnell ring* of G and denoted by $\mathcal{D}(G)$, so the Strong Atiyah Conjecture over $\mathbb{C}$ for a torsion-free group G is the statement that the Linnell ring of G is a division ring. In this case, the Linnell ring of G has the Linnell property defined above. Moreover, when G is torsion-free and satisfies the Strong Atiyah Conjecture over $\mathbb{C}$, then the usual L^2-Betti numbers of G can be computed using the Linnell division ring as follows:

$$b_n^{(2)}(G) \;=\; \dim_{\mathcal{D}(G)} \mathrm{H}_n(G;\mathcal{D}(G))$$

We refer the reader to [Lüc02, Chapter 10] for more details on this material.

When G is locally indicable, then G satisfies the strong Atiyah Conjecture over $\mathbb{C}$ by a result of Jaikin-Zapirain and López-Álvarez [JZLÁ20]. We emphasise the following corollary of their result.

Theorem 2.5.3 ([JZLÁ20, Corollary 1.4])**.** *Let G be locally indicable and let k be a field of characteristic zero. Then $\mathcal{D}_{k[G]}$ exists.*

2.5.2 Properties

We collect some properties of k-L^2-invariants.

Proposition 2.5.4. *Let G be a locally indicable group and k be a division ring such that $\mathcal{D}_{k[G]}$ exists.*

(i) *If $H \leqslant G$ is a subgroup of finite index, then $b_n^{(2)}(H;k) = [G:H]b_n^{(2)}(G;k)$ for all $n \geqslant 0$.*
(ii) *If G is of finite type, then $\chi(G) = \sum_{n\geqslant 0}(-1)^n b_n^{(2)}(G;k)$.*
(iii) *For all $n \geqslant 0$, we have $b_n^{(2)}(G;k) = b^n_{(2)}(G;k)$.*

Proof. Fix a projective resolution $P_\bullet \to k$. By the Linnell property, $P_\bullet \otimes_{k[G]} \mathcal{D}_{k[G]}$ and $P_\bullet \otimes_{k[H]} \mathcal{D}_{k[H]}$ are isomorphic as chain complexes of $\mathcal{D}_{k[H]}$-modules. Hence,

$$\begin{aligned} b_n^{(2)}(H;k) &= \dim_{\mathcal{D}_{k[H]}} \mathrm{H}_n(P_\bullet \otimes_{k[H]} \mathcal{D}_{k[H]}) \\ &= \dim_{\mathcal{D}_{k[H]}} \mathrm{H}_n(P_\bullet \otimes_{k[G]} \mathcal{D}_{k[G]}) \\ &= [G:H]b_n^{(2)}(G;k), \end{aligned}$$

since $\dim_{\mathcal{D}_{k[H]}} \mathcal{D}_{k[G]} = [G:H]$ by the Linnell property, proving (i). Claim (ii) follows immediately from the rank-nullity theorem and a standard dimension counting argument (compare the usual proof that the Euler characteristic is the alternating sum of Betti numbers). Finally, claim (iii) follows at once from the familiar isomorphism

$$\mathrm{Hom}_{\mathcal{D}_{k[G]}}(\mathrm{H}_n(G;\mathcal{D}_{k[G]}), \mathcal{D}_{k[G]}) \cong \mathrm{H}^n(G;\mathcal{D}_{k[G]}).$$ □

If G has a finite-index subgroup H such that $\mathcal{D}_{k[H]}$ exists, then item (i) allows us to define $b_n^{(2)}(G;k) := \frac{b_n^{(2)}(H;k)}{[G:H]}$.

The next property gives a useful vanishing criterion for k-L^2-Betti numbers. It is an analogue of [Gab02, Théorème 6.6] for k-L^2-Betti numbers. When the quotient is $\mathbb{Z}$, the author obtained this result in [Fis24a, Theorem 6.4]. The proof in the case where the quotient is amenable is completely analogous, and was originally proven in the article [FK24] of the author and Klinge.

Theorem 2.5.5. *Let $1 \to N \to G \to Q \to 1$ be a short exact sequence of groups, where G is locally indicable, Q is infinite amenable, k is a field, and $\mathcal{D}_{k[G]}$ exists. If $b_n^{(2)}(N;k) < \infty$, then $b_n^{(2)}(G) = 0$.*

Proof. Let $C_\bullet \to k \to 0$ be a free resolution of $k[G]$-modules. Since $\mathcal{D}_{k[G]}$ is Linnell, using [Tam54] we have identifications $\mathcal{D}_{k[G]} = \operatorname{Ore}(\mathcal{D}_{k[N]} * Q)$ and

$$\mathcal{D}_{k[N]} \otimes_{kN} C_n \cong \mathcal{D}_{k[N]} \otimes_{kN} \bigoplus_{I_n} k[G] \cong \bigoplus_{I_n} \mathcal{D}_{k[N]} \otimes_{kN} k[G] \cong \bigoplus_{I_n} \mathcal{D}_{k[N]} * Q$$

for some index set I_n, for each integer n. Thus, we have inclusions of chain complexes

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \bigoplus_{I_{n+1}} \mathcal{D}_{k[N]} * Q & \longrightarrow & \bigoplus_{I_n} \mathcal{D}_{k[N]} * Q & \longrightarrow & \bigoplus_{I_{n-1}} \mathcal{D}_{k[N]} * Q & \longrightarrow \cdots \\ & \big\downarrow & & \big\downarrow & & \big\downarrow & \\ \cdots \longrightarrow & \bigoplus_{I_{n+1}} \mathcal{D}_{k[G]} & \longrightarrow & \bigoplus_{I_n} \mathcal{D}_{k[G]} & \longrightarrow & \bigoplus_{I_{n-1}} \mathcal{D}_{k[G]} & \longrightarrow \cdots, \end{array}$$

where the upper chain complex computes $\mathrm{H}_\bullet(N; \mathcal{D}_{k[N]})$ and the lower chain complex computes $\mathrm{H}_\bullet(G; \mathcal{D}_{k[G]})$.

For any n-cycle $c \in Z_n(\bigoplus_{I_\bullet} \mathcal{D}_{k[G]})$, there is some $\alpha \in \mathcal{D}_{k[N]} * Q$ such that $\alpha c \in \bigoplus_{I_\bullet} \mathcal{D}_{k[N]} * Q$. Then $(\mathcal{D}_{k[N]} * Q) \cdot \alpha c$ is an infinite-dimensional $\mathcal{D}_{k[N]}$-subspace of $Z_n(\bigoplus_{I_\bullet} \mathcal{D}_{k[N]} * Q)$. But $b_n^{(2)}(N;k) < \infty$, so there must be some $d \in \bigoplus_{I_{n+1}} \mathcal{D}_{k[N]} * Q$ such that $\partial d = \beta \cdot \alpha c$ for some nonzero $\beta \in \mathcal{D}_{k[N]} * Q$. Hence, $\partial((\beta\alpha)^{-1} d) = c$, which proves that $b_n^{(2)}(G;k) = 0$. □

We will also make use of the following result, which states that the vanishing of the top-degree k-L^2-Betti number passes to subgroups. This was proved in the article [FM23] of Morales and the author.

Proposition 2.5.6. *Let G be a locally indicable group with $\operatorname{cd}_k(G) = n$ for some division ring k and suppose that $\mathcal{D}_{k[G]}$ exists. If $b_n^{(2)}(G;k) = 0$, then $b_n^{(2)}(H;k) = 0$ for every subgroup $H \leqslant G$.*

Proof. We claim that the natural map

$$\mathcal{D}_{k[H]} \otimes_{k[H]} P \to \mathcal{D}_{k[G]} \otimes_{k[G]} P$$

is injective for any projective $k[G]$-module P. Since there is a module Q such that $P \oplus Q$ is free, it is enough to prove the case where P is free. It then suffices to prove the case $P = k[G]$. But then the claim is just a restatement of the Linnell condition, which Hughes-free division rings possess.

The result now follows quickly. Let $0 \to P_n \to \cdots \to P_0 \to k \to 0$ is a projective resolution of the trivial $k[G]$-module k. Then

$$\mathrm{H}_n^{(2)}(H; k) = \ker(\mathcal{D}_{k[H]} \otimes_{k[H]} P_n \to \mathcal{D}_{k[H]} \otimes_{k[H]} P_{n-1})$$

injects into

$$\mathrm{H}_n^{(2)}(G; k) = \ker(\mathcal{D}_{k[G]} \otimes_{k[G]} P_n \to \mathcal{D}_{k[G]} \otimes_{k[G]} P_{n-1}),$$

by the previous paragraph. □

Chapter 3

Novikov homology and algebraic fibrings of RFRS groups

The purpose of this chapter is to establish Sikorav's Theorem over arbitrary coefficient rings, and then use it to prove a theorem on virtual algebraic fibrations of RFRS groups.

3.1 The Σ-invariant and Novikov homology

The material from this section is based on Sections 3, 4, and 5 of [Fis24a].

3.1.1 Valuations on free resolutions

In this section, we introduce valuations on free resolutions over a group ring. We will be very closely following Bieri and Renz [BR88] where the theory is developed in the case where the ring is $\mathbb{Z}$. Their proofs go through without change when $\mathbb{Z}$ is replaced by an arbitrary ring R.

Let R be a ring, G a group, and M a left $R[G]$-module. Recall that a *free resolution* of M is an exact sequence

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

of left $R[G]$-modules, where F_i is free for all $i \geqslant 0$. The *boundary maps* of the resolution are denoted $\partial_n \colon F_n \to F_{n-1}$, though we will often omit the subscript. The free resolution is usually denoted by $F_\bullet \to M$. Let F be the free $R[G]$-module $\bigoplus_{i=0}^{\infty} F_i$, and define the *$n$-skeleton* of F to be $F^{(n)} := \bigoplus_{i=0}^{n} F_i$. The elements of F are called *chains*, so a chain is not necessarily an element of F_i for any i in our context. Fixing a basis X_i for each F_i, we note that $X := \bigcup_{i=0}^{\infty} X_i$ is a basis for F and $X^{(n)} := \bigcup_{i=0}^{n} X_i$ is a basis for $F^{(n)}$. The resolution $F_\bullet \to M$ is *admissible with respect to X* if $\partial x \neq 0$

for every $x \in X$. We will always assume that our free resolutions are admissible with respect to the basis we are working with. This is not a strong requirement, since if all boundary maps are nonzero, then F has a basis with respect to which $F_\bullet \to M$ is admissible; otherwise we can truncate the resolution and choose a basis to obtain an admissible resolution of finite length. We also define the *support* of a chain $c \in F$ (with respect to X), denoted $\operatorname{supp}_X(c)$, as follows: every chain $c \in F$ can be written uniquely as $\sum_{g \in G, x \in X} r_{g,x} gx$, where $r_{g,x} \in R$. Then $\operatorname{supp}_X(c) := \{gx : r_{g,x} \neq 0\}$; we will usually drop the subscript X when the basis is understood.

Let $\chi\colon G \to \mathbb{R}$ be a non-trivial *character*, that is, a nonzero group homomorphism from G to the additive group $\mathbb{R}$. This provides the elements of G with a notion of height, which we now extend to the chains of F. Let $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$, where ∞ is an element such that $t < \infty$ for every $t \in \mathbb{R}$. We construct a function $v_X\colon F \to \mathbb{R}_\infty$ via the following inductive procedure. For an element $c \in F_0$, define $v_X(c) = \inf\{\chi(g) : gx \in \operatorname{supp}(c)\}$. Let $n > 0$ and assume that we have defined v_X on F_{n-1}. For $x \in X_n$, let $v_X(x) := v_X(\partial x)$. For $c \in F_n$, set $v_X(c) = \inf\{\chi(g) + v_X(x) : gx \in \operatorname{supp}(c)\}$. For an arbitrary $c \in F$, write $c = \sum_i c_i$, where $c_i \in F_i$, and define $c = \inf_i\{v_X(c_i)\}$. We are assuming the convention $v_X(0) = \infty$, since $\operatorname{supp}(0) = \varnothing$. The function v_X is called the *valuation extending χ with respect to X*. It is clear from the definition that $v_X(c) = \inf\{\chi(g) + v_X(x) : gx \in \operatorname{supp}(c)\}$ for any chain $c \in F$. Again, we will usually drop the X in the subscript when the basis is understood.

Proposition 3.1.1. *For the valuation $v_X = v\colon F \to \mathbb{R}_\infty$ defined above and for any $c, c' \in F$ and $g \in G$, we have*

1. $v(c + c') \geqslant \min\{v(c), v(c')\}$;
2. $v(c) \leqslant v(rc)$ *for all* $r \in R$, *and* $v(c) = v(rc)$ *if* r *is not a zero-divisor;*
3. *if* $v(c) \neq v(c')$, *then* $v(c + c') = \min\{v(c), v(c')\}$;
4. $v(gc) = \chi(g) + v(c)$;
5. $v(c) = \infty$ *if and only if* $c = 0$;
6. *if* $c \in \bigoplus_{i \geqslant 1} F_i$, *then* $v(\partial c) \geqslant v(c)$.

Proof. (1) follows from the fact that $\operatorname{supp}(c + c') \subseteq \operatorname{supp}(c) \cup \operatorname{supp}(c')$. The first part of (2) follows from the fact that $\operatorname{supp}(c) \supseteq \operatorname{supp}(rc)$. If r is not a zero-divisor then $\operatorname{supp}(c) = \operatorname{supp}(rc)$, which yields the second statement of (2).

To prove (3), assume without loss of generality that $v(c) < v(c')$. Then,

$$v(c) = v((c + c') - c') \geqslant \min\{v(c + c'), v(-c')\} = \min\{v(c + c'), v(c')\}.$$

Then $\min\{v(c+c'), v(c')\} = v(c+c')$, since we assumed that $v(c) < v(c')$. Hence, $v(c) = \min\{v(c), v(c')\} \geqslant v(c+c')$. But $v(c+c') \geqslant \min\{v(c), v(c')\}$ by (1), so we obtain (3).

For (4), we have

$$\begin{aligned} v(gc) &= \inf\{\chi(h) + v(x) : hx \in \operatorname{supp}(gc)\} \\ &= \inf\{\chi(g(g^{-1}h)) + v(x) : hx \in g \cdot \operatorname{supp} c\} \\ &= \chi(g) + \inf\{\chi(g^{-1}h) + v(x) : (g^{-1}h)x \in \operatorname{supp} c\} \\ &= \chi(g) + v(c). \end{aligned}$$

For (5), we first show that if $c \in F_n \setminus \{0\}$, then $v(c) < \infty$ by induction on n. This is true for $n = 0$ since $\chi(G) \subseteq \mathbb{R}$. Now let $n > 0$. Since $c \neq 0$, there is some element gx in its support, where $g \in G$ and $x \in X$. Then

$$v(c) \leqslant v(gx) = \chi(g) + v(x) = \chi(g) + v(\partial x) < \infty$$

by the inductive hypothesis and by admissibility of $F_\bullet \to M$ with respect to X. For a general nonzero element $c \in F$, write $c = \sum_i c_i$ with $c_i \in F_i$. Then $v(c) = \inf_i\{v(c_i)\} < \infty$ since at least one of the chains c_i is nonzero. Conversely, if $c = 0$, then $v(c) = \infty$, since the infimum of the empty set is ∞.

For (6), let $c = \sum_{g\in G, x\in X} r_{g,x} gx \in \bigoplus_{i\geqslant 1} F_i$. Then

$$\begin{aligned} v(\partial c) &= v\left(\partial\left(\sum_{g\in G, x\in X} r_{g,x} gx\right)\right) \\ &= v\left(\sum_{g\in G, x\in X} r_{g,x} g\partial x\right) \\ &\geqslant \inf\{v(r_{g,x} g\partial x) : gx \in \operatorname{supp}(c)\} && \text{(by (1))} \\ &\geqslant \inf\{v(g\partial x) : gx \in \operatorname{supp}(c)\} && \text{(by (2))} \\ &= \inf\{\chi(g) + v(\partial x) : gx \in \operatorname{supp}(c)\} \\ &= \inf\{\chi(g) + v(x) : gx \in \operatorname{supp}(c)\} \\ &= v(c). \end{aligned}$$

□

Definition 3.1.2 (Valuation subcomplex and essential acyclicity)**.** Given an admissible free resolution $F_\bullet \to M$ over $R[G]$ (with respect to some fixed basis X), a non-trivial character $\chi\colon G \to \mathbb{R}$, and the valuation $v\colon F \to \mathbb{R}_\infty$ extending χ, define the *valuation subcomplex* of F with respect to v to be the chain complex $\cdots \to F_n^v \to \cdots \to F_0^v \to M \to 0$, where $F_n^v = \{c \in F_n : v(c) \geqslant 0\}$. We denote the valuation subcomplex by $F^v_\bullet \to M$ and let $F^v := \bigoplus_{i=0}^{\infty} F^v_i$. Proposition 3.1.1(6) ensures that $F^v_\bullet \to M$ is a chain complex of left $R[G_\chi]$-modules, where G_χ is the monoid $\{g \in G : \chi(g) \geqslant 0\}$. It is not hard to show that each F^v_i is a free $R[G_\chi]$-module and has an $R[G_\chi]$-basis of cardinality $|X_i|$, where X_i is an $R[G]$-basis for F_i.

The chain complex $F^v_\bullet \to M$ is *essentially acyclic in dimension* n if there is a real number $D \geqslant 0$ such that for every cycle $z \in F^v_n$ there is a $c \in F_{n+1}$ with $\partial c = z$ and $D \geqslant v(z) - v(c)$. We extend the definition of essential acyclicity to dimension -1 by declaring that $v(m) = 0$ for all $m \in M \setminus \{0\}$.

The definition of essential acyclicity in dimension n is equivalent to the following seemingly weaker condition: for every cycle $z \in F^v_n$, there is a $c \in F_{n+1}$ such that $\partial c = z$ and $v(c) \geqslant -D$. To see this, let $z \in F^v_n$ be a cycle. It is easily shown that $v(F) \subseteq \chi(G) \cup \{\infty\}$, so there is a $g \in G$ such that $\chi(g) = v(z)$. Since $g^{-1}z$ is also in F^v_n with $v(g^{-1}z) = 0$, there is some $c \in F^v_{n+1}$ such that $\partial c = g^{-1}z$ and $v(c) \geqslant -D$. Thus, $\partial(gc) = z$, and $D \geqslant v(z) - v(gc)$.

3.1.2 Horochains

Definition 3.1.3 (The complex of horochains and horo-acyclicity)**.** Let $F_\bullet \to M$ be an admissible free resolution with respect to some basis X, let $\chi\colon G \to \mathbb{R}$ be a non-trivial character, and let $v\colon F \to \mathbb{R}_\infty$ be the valuation extending χ. Define $\widehat{F}$ to be the left $R[G]$-module of formal series that are finitely supported below every height. More precisely, $\widehat{F}$ is the $R[G]$-module of formal series $\sum_{g \in G, x \in X} r_{g,x} gx$ such that

$$\{gx : v(gx) \leqslant t, r_{g,x} \neq 0\}$$

is finite for every $t \in \mathbb{R}$. The elements of $\widehat{F}$ are called *horochains*. If $\hat{c} \in \widehat{F}$, then its *support* is $\operatorname{supp}_X(\hat{c}) := \{gx : r_{g,x} \neq 0\}$. Let $\widehat{F}_i \subseteq \widehat{F}$ be the subset of horochains with support in F_i and let $\widehat{F}^{(n)} := \bigoplus_{i=0}^{n} \widehat{F}_i$. Proposition 3.1.1(6) guarantees that $\partial\colon F_n \to F_{n-1}$ extends to a map $\partial\colon \widehat{F}_n \to \widehat{F}_{n-1}$ in the obvious way, so we get a complex $\cdots \to \widehat{F}_n \to \cdots \to \widehat{F}_0 \to 0$. Note that $\widehat{F}$ is not equal to $\bigoplus_{i=0}^{\infty} \widehat{F}_i$ since the support of a horochain might intersect infinitely many of the modules $\widehat{F}_i$. A cycle in the chain complex $\widehat{F}$ is called a *horocycle*. We say that $F_\bullet \to M$ is *horo-acyclic* in dimensions $n \geqslant 0$ with respect to v if the chain complex $\cdots \to \widehat{F}_1 \to \widehat{F}_0 \to 0$ is acyclic in dimension n.

We can extend the definition of v to $\widehat{F}$ by defining $v(\hat{c}) := \inf\{v(gx) : \operatorname{supp}(\hat{c})\}$ for any horochain $\hat{c}$. If $\hat{c} \neq 0$, then $\{v(gx) : \operatorname{supp}(\hat{c})\}$ is nonempty and attains a minimum because chains are finitely supported below any given height. Properties (1) through (5) of Proposition 3.1.1 hold in this setting with the same proofs.

A version of Proposition 3.1.1(6) holds for horochains, namely we have $v(\partial\hat{c}) \geqslant v(\hat{c})$ for all horochains $\hat{c}$, but we need to modify the proof: If $\hat{c} = 0$, then the claim is clear. Otherwise, let $\hat{c} \neq 0$ be a horochain, and let $gx \in \operatorname{supp}(\hat{c})$ be such that $v(gx) = v(\hat{c})$. By the finite version of (6), we have that $v(\partial g'x') \geqslant v(g'x') \geqslant v(gx)$ for every $g'x' \in \operatorname{supp}(\hat{c})$. Since every $g''x'' \in \operatorname{supp}(\partial\hat{c})$ is contained in $\operatorname{supp}(\partial g'x')$ for some $g'x' \in \operatorname{supp}(\hat{c})$, we have that $v(g''x'') \geqslant v(\partial g'x') \geqslant v(gx)$ for every $g''x'' \in \operatorname{supp}(\partial\hat{c})$. Thus, $v(\partial\hat{c}) \geqslant v(\hat{c})$.

The following lemma will be used in the proof of Theorem 3.1.7.

Lemma 3.1.4. *Let $F_\bullet \to M$ (resp. $F'_\bullet \to M$) be a free resolution over $R[G]$ admissible with respect to a basis X (resp. X'), and let v (resp. v') be the valuation extending a non-trivial character $\chi\colon G \to \mathbb{R}$. Suppose that $F^{(n)}$ is finitely generated, and that $\varphi\colon F \to F'$ is a homomorphism of $R[G]$-modules. Then*

(1) *φ induces a homomorphism of left $R[G]$-modules given by*

$$\widehat{\varphi}\colon \widehat{F}^{(n)} \to \widehat{F}', \ \sum r_{g,x}gx \mapsto \sum r_{g,x}g\varphi(x) \quad ;$$

(2) *$v'(\widehat{\varphi}(\hat{c})) \geqslant v(\hat{c}) + \min_{x\in X^{(n)}}\{v'(\varphi(x)) - v(x)\}$ for every $\hat{c} \in F^{(n)}$.*

Proof. For (1), we need to show that $\widehat{\varphi}(\hat{c})$ is a horochain for any horochain $\hat{c} \in \widehat{F}^{(n)}$. To this end, let $\hat{c} = \sum r_{g,x}gx$, and note that there are only finitely many elements $x \in X$ such that $gx \in \operatorname{supp}_X(\hat{c})$. If $\widehat{\varphi}(\hat{c})$ is not a horochain, then the set $\{gx \in \operatorname{supp}_X(\hat{c}) : v'(g\varphi(x)) \leqslant t\}$ is infinite for some $t \in \mathbb{R}$. Since $F^{(n)}$ is finitely generated, there is some fixed $y \in X$ such that $v'(g\varphi(y)) \leqslant t$ and $gy \in \operatorname{supp}_X(\hat{c})$ for infinitely many values of $g \in G$. But then

$$\begin{aligned} v(gy) &= \chi(g) + v(y) \\ &= \chi(g) + v'(\varphi(y)) + v(y) - v'(\varphi(y)) \\ &= v'(g\varphi(y)) + v(y) - v'(\varphi(y)) \\ &\leqslant t + v(y) - v'(\varphi(y)) \end{aligned}$$

for infinitely many $gy \in \operatorname{supp}_X(\hat{c})$, but $\hat{c}$ is a horochain.

For (2), write $\hat{c} = \sum_{x\in X^{(n)}} \hat{c}_x$, where $\hat{c}_x = \sum_{g\in G} r_{g,x} gx$. Then

$$\begin{aligned}
v'(\widehat{\varphi}(\hat{c})) &\geqslant \min_{x\in X^{(n)}} \{v'(\widehat{\varphi}(\hat{c}_x))\} \\
&\geqslant \min_{x\in X^{(n)}} \{\inf\{v'(g\varphi(x)) : gx \in \operatorname{supp} \hat{c}_x\}\} \\
&= \min_{x\in X^{(n)}} \{\inf\{v(gx) : gx \in \operatorname{supp} \hat{c}_x\} + v'(\varphi(x)) - v(x)\} \\
&= \min_{x\in X^{(n)}} \{v(\hat{c}_x) + v'(\varphi(x)) - v(x)\} \\
&\geqslant \min_{x\in X^{(n)}} \{v(\hat{c}_x)\} + \min_{x\in X^{(n)}} \{v'(\varphi(x)) - v(x)\} \\
&= v(\hat{c}) + \min_{x\in X^{(n)}} \{v'(\varphi(x)) - v(x)\}. \qquad \square
\end{aligned}$$

Note that Lemma 3.1.4(2) applies to chains in F, since these are just finite horochains. We will use this in the proof Theorem 3.1.7.

3.1.3 Characterisations of the Σ-invariant

We introduce the invariants $\Sigma^n_R(G; M)$, which are generalisations of the classical Bieri–Neumann–Strebel invariant [BNS87] and its higher dimensional analogues [BR88]. The only difference is that we work over a general ring R, while the higher BNS invariants are defined over $\mathbb{Z}$.

Let G be a group. We declare two characters $\chi, \chi' \colon G \to \mathbb{R}$ to be *equivalent* if $\chi = \alpha \cdot \chi'$ for some $\alpha > 0$ and let $S(G)$ denote the set of equivalence classes of nonzero characters. We call $S(G)$ the *character sphere* of G, because it can be given the topology of a sphere when G is finitely generated.

Definition 3.1.5 (Σ-invariants)**.** Let M be an $R[G]$-module. Then define

$$\Sigma^n_R(G; M) = \{[\chi] \in S(G) : M \in \mathtt{FP}_n(R[G_\chi])\},$$

where $G_\chi = \{g \in G : \chi(g) \geqslant 0\}$. Note that $G_\chi = G_{\chi'}$ if $[\chi] = [\chi']$, so $\Sigma^n_R(G; M)$ is well-defined.

Definition 3.1.6 (Novikov ring)**.** Let G be a group, let R be a ring, and let $\chi \colon G \to \mathbb{R}$ be a character. Then the *Novikov ring* $\widehat{R[G]}^\chi$ is the set of formal sums

$$\sum_{g\in G} r_g g$$

such that $\{g \in G : r_g \neq 0 \text{ and } \varphi(g) \leqslant t\}$ is finite for every $t \in \mathbb{R}$. We give $\widehat{R[G]}^\chi$ a ring structure by defining $rg + r'g := (r + r')g$ and $rg \cdot r'g' := rr'gg'$ for $r, r' \in R$, $g, g' \in G$, and extending multiplication to all of $\widehat{R[G]}^\chi$ in the obvious way.

Theorem 3.1.7 gives several characterisations of the Σ-invariants. More specifically, we will need the characterisation of $\Sigma^n_R(G;M)$ in terms of the vanishing of Novikov homology; this is the equivalence of (1) and (5) in the following theorem, which should be thought of as a higher dimensional version of Sikorav's theorem [Sik87].

Theorem 3.1.7. *Let R be a ring, let M be a left $R[G]$-module of type $\mathtt{FP}_n$, and let $\chi\colon G \to \mathbb{R}$ be a non-trivial character. Let $F_\bullet \to M$ be a free resolution admissible with respect to a basis $X = \bigcup_{i=0}^{\infty} X_i$ and with finitely generated n-skeleton $F^{(n)}$. Let $v\colon F \to \mathbb{R}_\infty$ be the valuation extending χ with respect to X. The following are equivalent:*

1. *$[\chi] \in \Sigma^n_R(G;M)$;*
2. *$F^v_\bullet \to M$ is essentially acyclic in dimensions $-1,\dots,n-1$;*
3. *there is a chain map $\varphi\colon F \to F$ lifting the identity id_M such that $v(\varphi(c)) > v(c)$ for every $c \in F^{(n)}$;*
4. *$F_\bullet \to M$ is horo-acyclic in dimensions $0,\dots,n$ with respect to v;*
5. *$\mathrm{Tor}_i^{R[G]}(\widehat{R[G]}^{\chi}, M) = 0$ for all $0 \leqslant i \leqslant n$.*

The strategy of the proof will be as follows: we begin by proving (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (2). This is done by Schweitzer in the appendix of [Bie07] in the case $R = \mathbb{Z}$. Once this is done, we prove the equivalence of (4) and (5), again following Schweitzer. Finally, we prove the equivalence of (1) and (2) following the appendix to Theorem 3.2 in [BR88], where again this is done in the case $R = \mathbb{Z}$. The proofs below are essentially the same as those given in the references just cited; there is no crucial dependence on the coefficient ring R.

Proof of (2) $\Rightarrow$ (3). Assume that $F^v_\bullet \to M$ is essentially acyclic in dimensions $\leqslant n-1$ and let $D > 0$ be a constant such that for each $k < n$ and every cycle $z \in F_k$, there is a chain $c \in F_{k+1}$ with $\partial c = z$ and $D \geqslant v(z) - v(c)$. We will construct a chain map $\varphi\colon F \to F$ lifting id_M such that $v(\varphi(c)) > v(c) + (n-k)D$ for every $c \in F^{(k)}$, which implies (3).

We define φ on $F^{(k)}$ by induction on k. For the base case, let $x \in X_0$ be arbitrary, and fix some $g \in G$ such that $\chi(g) > (n+1)D$. The element $g^{-1}\partial x \in M$ is a cycle, so there is some $c_x \in F_0$ such that $\partial c_x = g^{-1}\partial x$ and $D \geqslant v(g^{-1}\partial x) - v(c_x) = -v(c_x)$, since $v|_{M\setminus\{0\}} = 0$. Define φ on $F^{(0)}$ by setting $\varphi(x) = gc_x$ for each $x \in X_0$. It is clear that $\mathrm{id}_M\partial = \partial\varphi$ on $F^{(0)}$. By Lemma 3.1.4(2),

$$v(\varphi(c)) \geqslant v(c) + \min_{x\in X_0}\{v(\varphi(x)) - v(x)\} > v(c) + nD$$

for every $c \in F^{(0)}$.

Let $k > 0$ and suppose φ is defined on $F^{(k-1)}$ such that it lifts id_M and $v(\varphi(c)) > v(c) + (n-k+1)D$ for all $c \in F^{(k-1)}$. Let $x \in X_k$ and note that $\varphi(\partial x)$ is a cycle. By essential acyclicity, there is a chain $d_x \in F_k$ such that $\partial d_x = \varphi(\partial x)$ and $D \geqslant v(\varphi(\partial x)) - v(d_x)$. Define φ on $F^{(k)}$ by setting $\varphi(x) = d_x$. Then $\varphi\partial = \partial\varphi$ by construction, and for every $x \in X_k$ we have

$$\begin{aligned} v(\varphi(x)) - v(x) &= v(d_x) - v(x) \\ &\geqslant v(\varphi(\partial x)) - v(x) - D \\ &= v(\varphi(\partial x)) - v(\partial x) - D \\ &> (n-k)D \end{aligned}$$

by induction. By Lemma 3.1.4(2), we have

$$v(\varphi(c)) \geqslant v(c) + \min_{x \in X_k}\{v(\varphi(x)) - v(x)\} > v(c) + (n-k)D. \qquad \square$$

We pause here to prove a lemma that will immediately imply (3) ⇒ (4) and will be useful in the proofs of (4) ⇒ (2) and (4) ⇒ (5). We recall that the maps $\widehat{H}$ and $\widehat{\varphi}$ that appear in the statement of the lemma below are defined in Lemma 3.1.4.

Lemma 3.1.8. *With the assumptions of Theorem 3.1.7, let $\varphi\colon F \to F$ be a chain map lifting id_M such that $v(\varphi(c)) > v(c)$ for all $c \in F^{(n)}$ and let $H\colon F \to F$ be a chain homotopy such that $\partial H + H\partial = \mathrm{id}_F - \varphi$. Let $\hat{z} \in \widehat{F}^{(n)}$ be a horocycle and define $\hat{c}_{\hat{z}} := \sum_{i=0}^{\infty} \widehat{H}\widehat{\varphi}^i(\hat{z})$. Then $\hat{c}_{\hat{z}}$ is a horochain and $\partial\hat{c}_{\hat{z}} = \hat{z}$.*

Proof. By Lemma 3.1.4(2) there are constants α and β such that

$$v(\widehat{\varphi}(\hat{c})) \geqslant v(\hat{c}) + \alpha \quad \text{and} \quad v(\widehat{H}(\hat{c})) \geqslant v(\hat{c}) + \beta$$

for every horochain $\hat{c} \in \widehat{F}^{(n)}$. Moreover, $\alpha > 0$ since $v(\varphi(f)) > v(f)$ for every $f \in F^{(n)}$. To see that $\hat{c}$ is a horochain, by induction we have $v(\widehat{H}\widehat{\varphi}^i(\hat{z})) \geqslant v(\hat{z}) + i\alpha + \beta$, so for all $t \in \mathbb{R}$ there are only finitely many integers $i \geqslant 0$ such that $v(\widehat{H}\widehat{\varphi}^i(\hat{z})) \leqslant t$. Since $\mathrm{supp}(\hat{c}_{\hat{z}}) \subseteq \bigcup_{i=0}^{\infty} \mathrm{supp}(\widehat{H}\widehat{\varphi}^i(\hat{z}))$ and each $\widehat{H}\widehat{\varphi}^i(\hat{z})$ is a horochain, it follows that there are only finitely many $gx \in \mathrm{supp}\,\hat{c}$ such that $v(gx) \leqslant t$, so $\hat{c}$ is a horochain.

Finally, we have

$$\partial\hat{c} = \sum_{i=0}^{\infty} \partial\widehat{H}\widehat{\varphi}^i(\hat{z}) = \sum_{i=0}^{\infty} (\mathrm{id}_{\widehat{F}^{(n)}} - \widehat{\varphi} - \widehat{H}\partial)\widehat{\varphi}^i(\hat{z}) = \sum_{i=0}^{\infty} (\widehat{\varphi}^i - \widehat{\varphi}^{i+1})(\hat{z}) = \hat{z}. \qquad \square$$

Proof of (3) ⇒ (4). By [Bro94, Lemma I.7.4], there is a chain homotopy $H\colon F \to F$ such that $\partial H + H\partial = \mathrm{id}_F - \varphi$. If $\hat{z} \in \widehat{F}^{(n)}$ is a horocycle, then $\partial \hat{c}_{\hat{z}} = \hat{z}$ by Lemma 3.1.8. □

Proof of (4) ⇒ (2). We will prove that $F^v_\bullet \to M \to 0$ is essentially acyclic in dimension k for all $k < n$ by induction on k. For the base case, we show that $F^v_\bullet \to M \to 0$ is exact at M, which implies essential acyclicity in dimension -1. Let $m \in M$. By exactness of $F_\bullet \to M \to 0$, there is a chain $c \in F_0$ such that $\partial c = m$. By horo-acyclicity in dimension 0, there is some horochain $\hat{c} \in \widehat{F}_1$ such that $\partial \hat{c} = c$. There are $c_- \in F_1$ and $\hat{c}_+ \in \widehat{F}_1$ such that $\hat{c} = c_- + \hat{c}_+$, where $v(c_-) < 0$ and $v(\hat{c}_+) \geqslant 0$. Then $\partial(c - \partial c_-) = m$ and

$$v(c - \partial c_-) = v(c - \partial(\hat{c} - \hat{c}_+)) = v(\partial \hat{c}_+) \geqslant v(\hat{c}_+) \geqslant 0.$$

This shows that $c - \partial c_0 \in F^v_0$, which proves that $F^v_\bullet \to M$ is exact at M.

Let $k > -1$ and suppose that $F^v_\bullet \to M$ is essentially acyclic in dimensions $< k$. By (2) ⇒ (3) applied at $k-1$, there is a chain map $\varphi\colon F \to F$ lifting id_M such that $v(\varphi(c)) > v(c)$ for all $c \in F^{(k)}$. Since id_F and φ both lift id_M and $F_\bullet \to M$ is acyclic, there is a chain homotopy $H\colon F \to F$ such that $\partial H + H\partial = \mathrm{id}_F - \varphi$ (see [Bro94, Lemma I.7.4]). As in the proof of Lemma 3.1.8, there are constants $\alpha > 0$ and $\beta < 0$ such that

$$v(\varphi(c)) \geqslant v(c) + \alpha \quad \text{and} \quad v(H(c)) \geqslant v(c) + \beta$$

for every $c \in F^{(k)}$.

Let $z \in F^v_k$ be a cycle. Since $F_\bullet \to M$ is acyclic, there is some $d \in F_{k+1}$ such that $\partial d = z$. Consider the horocycle $\hat{z} := d - \hat{d}_z$, where $\hat{d}_z = \sum_{i=0}^{\infty} H\varphi^i(z)$ is defined as in Lemma 3.1.8. Note that

$$v(H\varphi^i(z)) \geqslant v(z) + i\alpha + \beta \geqslant \beta$$

for every $i \geqslant 0$, and therefore that $v(\hat{d}_z) \geqslant \beta$. By horo-acyclicity in dimension $k+1$, there is a $(k+2)$-horochain $\hat{d}$ such that $\partial \hat{d} = \hat{z}$. As in the base case, there are $d_- \in F_{k+2}$ and $\hat{d}_+ \in \widehat{F}_{k+2}$ such that $\hat{d} = d_- + \hat{d}_+$, where $v(d_-) < 0$ and $v(\hat{d}_+) \geqslant 0$. Then $\partial(d - \partial d_-) = \partial d = z$, and

$$\begin{aligned} v(d - \partial d_-) &= v(\hat{d}_z + \hat{z} - \partial(\hat{d} - \hat{d}_+)) \\ &= v(\hat{d}_z + \partial \hat{d}_+) \\ &\geqslant \min\{v(\hat{d}_z), v(\partial \hat{d}_+)\} \\ &\geqslant \beta \end{aligned}$$

since $v(\hat{d}_z) \geqslant \beta$ and $v(\partial d_\infty) \geqslant v(d_\infty) \geqslant 0 > \beta$. Letting $D = -\beta$ in the definition of essential acyclicity, we see that $F^v_\bullet \to M$ is essentially acyclic in dimension k. □

Proof of (5) $\Rightarrow$ (4). Suppose that $\mathrm{Tor}_i^{R[G]}(\widehat{R[G]}^\chi, M) = 0$ for $0 \leqslant i \leqslant n$. Consider the chain map

$$\psi\colon \widehat{R[G]}^\chi \otimes_{R[G]} F \to \widehat{F}, \quad \alpha \otimes c \mapsto \alpha c$$

of left $R[G]$-modules. It is clear that ψ is injective. We claim that ψ induces an isomorphism $\widehat{R[G]}^\chi \otimes_{R[G]} F^{(n)} \to \widehat{F}^{(n)}$. To see this, simply note that for an arbitrary horochain

$$\hat{c} = \sum_{\substack{g\in G \\ x \in X^{(n)}}} r_{g,x} g x$$

in $\widehat{F}^{(n)}$, we have

$$\sum_{x\in X^{(n)}} \left(\sum_{g \in G} r_{g,x} g \right) \otimes x \overset{\varphi}{\mapsto} \hat{c}.$$

The horochain condition implies that the sums $\sum_{g\in G} r_{g,x} g$ are elements of $\widehat{R[G]}^\chi$. Thus, ψ is surjective on the n-skeleta and is therefore an isomorphism. Note that this only works because $X^{(n)}$ is finite; in general, we cannot expect ψ to be surjective since the support of a horochain might intersect infinitely many of the modules F_n. Since $\mathrm{Tor}_i^{R[G]}(\widehat{R[G]}^\chi \otimes_{R[G]} F, M) = 0$ for all $0 \leqslant i \leqslant n$, we conclude that $\mathrm{Tor}_i(\widehat{F}, M) = 0$ for $0 \leqslant i \leqslant n$ as well. □

Proof of (4) $\Rightarrow$ (5). The map ψ defined above is an isomorphism of the n-skeleta, so we immediately have that $\mathrm{Tor}_i^{R[G]}(\widehat{R[G]}^\chi, M) = 0$ for $0 \leqslant i \leqslant n-1$. Since ψ is not necessarily surjective as a map of the $(n+1)$-skeleta, we must work harder to show that $\mathrm{Tor}_n^{R[G]}(\widehat{R[G]}^\chi, M) = 0$. Let $z \in \widehat{R[G]}^\chi \otimes_{R[G]} F_n$ be an n-cycle, and let $\hat{z} = \psi(z)$. Since we are assuming that (4) holds, we may also assume that (3) holds and use the horochain $\hat{c}_{\hat{z}}$ from Lemma 3.1.8. Since $\hat{c}_{\hat{z}} \in \widehat{H}(\widehat{F}_n)$, we have that $\hat{c}_{\hat{z}}$ is in the $\widehat{R[G]}^\chi$-submodule of $\widehat{F}_{n+1}$ generated by $\widehat{H}(X_n)$, and thus $\hat{c}_{\hat{z}} \in \operatorname{im}\psi$ since this is a finite set. Let $c \in \widehat{R[G]}^\chi \otimes_{R[G]} F_{n+1}$ such that $\psi(c) = \hat{c}_{\hat{z}}$. Then $\psi\partial(c) = \partial\psi(c) = \partial\hat{c}_{\hat{z}} = \hat{z}$. But ψ is injective, so $\partial c = z$, proving that $\mathrm{Tor}_n^{R[G]}(\widehat{R[G]}^\chi, M) = 0$. □

We pause again before proving the equivalence of (1) and (2) to prove another lemma.

Lemma 3.1.9. *Free $R[G]$-modules are flat over $R[G_\chi]$.*

Proof. It suffices to prove that $R[G]$ is flat as an $R[G_\chi]$-module, since the direct sum of flat modules is flat. To this end, let $\iota\colon M \hookrightarrow N$ be an injection of right $R[G_\chi]$-modules; our goal is to show that $\iota \otimes \mathrm{id}\colon M \otimes_{R[G_\chi]} R[G] \to N \otimes_{R[G_\chi]} R[G]$ is injective. Let $g \in G$ be such that $\chi(g) < 0$ and consider the left $R[G_\chi]$-module $R[G_\chi]g^k = \{\alpha g^k : \alpha \in R[G_\chi], k \in \mathbb{Z}\}$. The modules $R[G_\chi]g^k$ form a directed system with respect to the inclusion maps $R[G_\chi]g^k \hookrightarrow R[G_\chi]g^l$ for $k \leqslant l$ and the direct limit is $\varinjlim R[G_\chi]g^k \cong R[G]$.

There are left $R[G_\chi]$-module isomorphisms $R[G_\chi]g^k \to R[G_\chi]$ given by right multiplication by g^{-k}. Then $R[G_\chi]g^k$ is flat over $R[G_\chi]$, so $M \otimes_{R[G_\chi]} R[G_\chi]g^k \to N \otimes_{R[G_\chi]} R[G_\chi]g^k$ is injective for all $k \in \mathbb{Z}$. By exactness of the direct limit,

$$\varinjlim(M \otimes_{R[G_\chi]} R[G_\chi]g^k) \to \varinjlim(N \otimes_{R[G_\chi]} R[G_\chi]g^k)$$

is injective. Since the direct limit commutes with the tensor product, the previous line implies $\iota \otimes \mathrm{id}_M$ is injective. □

We now return to the proof of Theorem 3.1.7.

Proof of (1) $\Leftrightarrow$ (2). Let $g \in G$ be such that $\chi(g) < 0$ and let E_k be the left $R[G_\chi]$-module $g^k F^v$. We denote the chain complexes $F^v_\bullet \to M$ and $(E_k)_\bullet \to M$ by $\widetilde{F}^v$ and $\widetilde{E}_k$, respectively.

Essential acyclicity in dimension j is equivalent to the existence of an integer $D \geqslant 0$ such that the inclusion-induced homomorphism $H_j(\widetilde{E}_k) \to H_j(\widetilde{E}_{k+D})$ is the zero map for all $k \in \mathbb{N}$. This in turn is equivalent to $\varinjlim \prod_I H_j(\widetilde{E}_k) = 0$ for any index set I. Here, for fixed I and j, the powers $\prod_I H_j(\widetilde{E}_k)$ form a directed system with respect to the inclusion-induced maps $\prod_I H_j(\widetilde{E}_k) \to \prod_I H_j(\widetilde{E}_l)$ for $k \leqslant l$. Indeed, if $D \geqslant 0$ is such that $H_j(\widetilde{E}_k) \to H_j(\widetilde{E}_{k+D})$ is the zero map, it is clear that the direct limit will be zero. Conversely, let $I = Z_j(\widetilde{E}_0) = Z_j(\widetilde{F}^v)$ be the set of j-cycles of $\widetilde{F}^v$ and consider the element $([x])_{x\in I} \in \prod_I H_j(\widetilde{E}_0)$. Since the direct limit is zero, there is some $D \geqslant 0$ such that $([x])_{x\in I} = 0$ in $\prod_I H_j(\widetilde{E}_D)$, which means that $\widetilde{F}^v$ is essentially acyclic in dimension j.

There is a short exact sequence of chain complexes $0 \to M \to \widetilde{E}_k \to E_k \to 0$, where, by abuse of notation, M is a chain complex concentrated in dimension -1 and E_k is the chain complex $(E_k)_\bullet \to 0$ with $(E_k)_0$ in dimension 0. The long exact sequence in homology associated to the short exact sequence gives $H_j(\widetilde{E}_k) \cong H_j(E_k)$ for $j \geqslant 1$. The interesting part of the long exact sequence is

$$0 \to H_0(\widetilde{E}_k) \to H_0(E_k) \xrightarrow{\delta} M \to H_{-1}(\widetilde{E}_k) \to 0,$$

where δ is the connecting homomorphism. By exactness of the direct power and direct limit functors, the sequence

$$0 \to \varinjlim \prod_I H_0(\widetilde{E}_k) \to \varinjlim \prod_I H_0(E_k) \xrightarrow{\prod_I \delta} \prod_I M \to \varinjlim \prod_I H_{-1}(\widetilde{E}_k) \to 0$$

is exact. Then $\widetilde{F}^v$ is essentially acyclic in dimension 0 if and only if δ induces an injection $\varinjlim \prod_I H_0(E_k) \to \prod_I M$ for every I. Moreover, $\widetilde{F}^v$ is essentially acyclic in dimension -1 if and only if δ induces a surjection $\varinjlim \prod_I H_0(E_k) \to \prod_I M$ for every I.

By Lemma 3.1.9, $F_\bullet \to M$ is a flat resolution of M by left $R[G_\chi]$-modules, so

$$\operatorname{Tor}_j^{R[G_\chi]}\left(\prod_I R[G_\chi], M\right) = H_j\left(\left(\prod_I R[G_\chi]\right) \otimes_{R[G_\chi]} F\right),$$

and therefore

$$\operatorname{Tor}_j^{R[G_\chi]}\left(\prod_I R[G_\chi], M\right) = \varinjlim H_j\left(\left(\prod_I R[G_\chi]\right) \otimes_{R[G_\chi]} E_k\right),$$

as $F = \varinjlim E_k$ and direct limits commute with tensor products and homology. Since $(E_k)_j$ is a finitely generated free $R[G_\chi]$-module for $j \leqslant n$, we have $(\prod_I R[G_\chi]) \otimes_{R[G_\chi]} (E_k)_j \cong \prod_I (E_k)_j$. Hence, $\operatorname{Tor}_j^{R[G_\chi]}(\prod_I R[G_\chi], M) = \varinjlim H_j(\prod_I E_k)$ for $j < n$.

To summarise the work done above, we have $\widetilde{F}^v$ is essentially acyclic in dimensions $-1 \leqslant j < n$ if and only if

(a) $(\prod_I R[G_\chi]) \otimes_{R[G_\chi]} M \to \prod_I M$ is surjective if $n = 0$ and

(b) $(\prod_I R[G_\chi]) \otimes_{R[G_\chi]} M \to \prod_I M$ is an isomorphism and

$$\operatorname{Tor}_j^{R[G_\chi]}\left(\prod_I R[G_\chi], M\right)$$

vanishes for $1 \leqslant j < n$ otherwise.

Here we have used the general fact that $\operatorname{Tor}_0^R(A, B) \cong A \otimes_R B$. Together with Lemma 1.1 and Proposition 1.2 of [BE74], (a) and (b) are equivalent to M being of type $\mathtt{FP}_n(R[G_\chi])$. Thus, we conclude that $[\chi] \in \Sigma_R^m(G; M)$ if and only $\widetilde{F}^v$ is essentially acyclic in dimensions $j = -1, 0, 1, \ldots, n-1$. □

3.2 Virtual algebraic fibrations of RFRS groups

A group G *algebraically fibres*, or simply *fibres*, if there is an epimorphism $\varphi\colon G \to \mathbb{Z}$ with finitely generated kernel. We will mostly be concerned with groups that virtually fibre, i.e. that admit a fibred finite-index subgroup. If $\mathcal{P}$ is a finiteness property of groups, we will say that G admits a *virtual algebraic fibration φ with kernel of type $\mathcal{P}$* if there is a finite-index subgroup $H \leqslant G$ with an epimorphism $\varphi\colon H \to \mathbb{Z}$ whose kernel is of type $\mathcal{P}$.

The connection between algebraic fibring and the Σ-invariants is given by the following special case of a result of Bieri–Neumann–Strebel (for the $n = 1$ case) and Bieri–Renz (for the higher degree case).

Theorem 3.2.1 ([BNS87],[BR88])**.** *Let G be a group of type* $\mathrm{FP}(R)$ *for some ring R. The kernel of the epimorphism $\chi\colon G \to \mathbb{Z}$ is of type* $\mathrm{FP}_n(R)$ *if and only* $[\chi]$ *and* $[-\chi]$ *are contained in* $\Sigma^n_R(G;R)$*.*

We will also need the following result of Kielak, which connects the L^2- and Novikov homology of RFRS groups. Kielak originally proved the result over $\mathbb{Q}$; the statement over arbitrary division rings is implied by Jaikin-Zapirain's appendix to [JZ21]. See also [OS24] for a different proof.

Theorem 3.2.2 ([Kie20b, Theorem 5.2])**.** *Let G be a RFRS of type* $\mathrm{FP}_n(k)$ *for some division ring k. If $b^{(2)}_i(G;k) = 0$, then there is a finite-index subgroup $H \leqslant G$ and an antipodally symmetric open set $U \subseteq S(H)$ such that $\overline{U} \supseteq S(G)$ and* $\mathrm{H}_i(H; \widehat{k[H]}^{\chi}) = 0$ *for all $\chi \in U$.*

We are now ready to prove one of the main results of this thesis.

Theorem 3.2.3. *Let G be a RFRS group of type* $\mathrm{FP}_n(k)$ *for some division ring k and nonnegative integer n. Then G virtually algebraically fibres with kernel of type* $\mathrm{FP}_n(k)$ *if and only if $b^{(2)}_i(G;k) = 0$ for all $i \leqslant n$.*

Proof. It is clear that a group of type $\mathrm{FP}_n(k)$ has finite k-L^2-Betti numbers in degrees at most n. Thus, if G virtually algebraically fibres with kernel of type $\mathrm{FP}_n(k)$, then $b^{(2)}_i(G;k) = 0$ for all $i \leqslant n$ by Proposition 2.5.4(i) and Theorem 2.5.5.

Conversely, suppose that $b^{(2)}_i(G;k) = 0$ for all $i \leqslant n$. By Theorem 3.2.2, there is a finite-index subgroup $H \leqslant G$ and a map $\chi\colon H \to \mathbb{Z}$ such that $\mathrm{H}_i(H; \widehat{k[H]}^{\pm\chi}) = 0$. But then χ and $-\chi$ are in $\Sigma^n_k(G;k)$ by Theorem 3.1.7. Finally, by Theorem 3.2.1, this implies that $\ker(\chi)$ is of type $\mathrm{FP}_n(k)$, as desired. $\square$

As a first application, we restrict the class of amenable RFRS groups of finite type. This can be viewed as an extension of the fact that virtually polycyclic RFRS groups must be virtually Abelian (Corollary 2.4.5). This is also related to a conjecture of P. Kropholler, which predicts that amenable groups of finite type are virtually solvable.

Corollary 3.2.4. *Let G be an amenable RFRS group of type* $\mathrm{FP}(k)$ *for some division ring k. Then G is virtually Abelian.*

Proof. We will prove that G is virtually poly-$\mathbb{Z}$, which is enough by Corollary 2.4.5. We induct on $\mathrm{cd}_k(G)$, which is necessarily finite. If $\mathrm{cd}_k(G) = 0$, then G is trivial and there is nothing to show. Suppose that $\mathrm{cd}_k(G) = n > 0$. Note that G is k-L^2-acyclic (this follows from [Tam54] and is a special case of Theorem 2.5.5), and G admits a virtual map to $\mathbb{Z}$ with kernel N of type $\mathrm{FP}(k)$. By [Fel71, Theorem 2.4], this implies that $\mathrm{cd}_k(N) = n - 1$, and by induction we conclude that N is virtually poly-$\mathbb{Z}$. But then it is not hard to see that G is itself virtually poly-$\mathbb{Z}$. □

In a different direction, we obtain a family of hyperbolic manifolds of all odd-dimensions whose fundamental groups virtually algebraically fibre with kernels of type $\mathrm{FP}(\mathbb{Q})$. The kernels of these algebraic fibrations were shown to be non-hyperbolic by Kudlinska in [Kud23], which gave the first family of examples of hyperbolic groups with non-hyperbolic subgroups of type $\mathrm{FP}(\mathbb{Q})$ in all cohomological dimensions. Note that Italiano, Martelli, and Migliorini give an example of a hyperbolic group (of cohomological dimension 5) with a non-hyperbolic subgroup of finite type in [IMM23]. It was also remarked by Llosa-Isenrich–Martelli–Py in [IMP24] that in even dimensions, Theorem 3.2.3 yields examples of hyperbolic groups with subgroups of type $\mathrm{FP}_n(\mathbb{Q})$ but not $\mathrm{FP}_{n+1}(\mathbb{Q})$ for all n, solving a homological analogue of a question of Brady [Bra99]. Brady's original question was completely answered by Llosa-Isenrich and Py in [LP24], where they provide examples of hyperbolic groups with subgroups of type F_n but not F_{n+1} for all n.

Corollary 3.2.5. *Let $\Gamma < \mathrm{PO}(n,1)$ be a uniform arithmetic lattice of simplest type.*

1. *If n is odd, then Γ virtually algebraically fibres with kernel of type* $\mathrm{FP}(\mathbb{Q})$.
2. *If n is even, then Γ virtually algebraically fibres with kernel of type* $\mathrm{FP}_{\frac{n}{2}-1}(\mathbb{Q})$ *but not* $\mathrm{FP}_{\frac{n}{2}}(\mathbb{Q})$.

Proof. By [BHW11], Γ acts geometrically on a CAT(0) cube complex, and therefore Γ is virtually RFRS by [Ago13]. By [Dod79], the L^2-Betti numbers of a lattice in $\mathrm{PO}(n,1)$ are concentrated in its middle dimension, so the result then follows immediately from Theorem 3.2.3 and the fact that if a group G is of type $\mathrm{FP}_n(\mathbb{Q})$ and $\mathrm{cd}_{\mathbb{Q}}(G) \leqslant n$, then G is of type $\mathrm{FP}(\mathbb{Q})$. □

3.2.1 Kernels with finite Betti numbers

In [JZ21, Corollary 1.5], Jaikin-Zapirain shows that if a finitely generated RFRS group G admits a map to $\mathbb{Z}$ with kernel having finite first Betti number, then G is virtually fibred. Theorem 3.2.3 allows one to extend this phenomenon to higher degrees.

Theorem 3.2.6. *Let k be a division ring and let G be a RFRS group of type $\mathrm{FP}_n(k)$. The following are equivalent:*

(1) *G admits a virtual map onto $\mathbb{Z}$ with kernel of type $\mathrm{FP}_n(k)$;*

(2) *G admits a virtual map onto $\mathbb{Z}$ with kernel N satisfying $b_i(N;k) < \infty$ for all $i \leqslant n$.*

Proof. If G virtually algebraically fibres with kernel of type $\mathrm{FP}_n(k)$, then it is clear that the Betti numbers $b_i(N;k)$ are finite for all $i \leqslant n$.

By Theorem 3.2.3, to prove the converse it suffices to show that $b_i^{(2)}(G;k) = 0$ for all $i \leqslant n$. By multiplicativity of the k-L^2-Betti numbers, we may assume that G admits a map onto $\mathbb{Z}$ with kernel N satisfying $b_i(N;k) < \infty$ for all $i \leqslant n$. We write $k[\mathbb{Z}]$ for the group algebra $k[G/N]$. By Shapiro's Lemma, $\mathrm{H}_i(N;k) \cong \mathrm{H}_i(G;k[\mathbb{Z}])$ for all i. Let

$$\cdots \to k[G]^{d_i} \to \cdots \to k[G]^{d_0} \to k \to 0$$

be a free resolution of the trivial $k[G]$-module k, where d_i is some cardinal for each i and d_i is finite for each $i \leqslant n$, and we use the notation $k[G]^{d_i}$ to denote the d_i-fold direct sum of copies of $k[G]$, as opposed to the d_i-fold direct product. The quotient map $G \to \mathbb{Z}$ induces a chain map

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\partial_{n+2}} & k[G]^{d_{n+1}} & \xrightarrow{\partial_{n+1}} & k[G]^{d_n} & \xrightarrow{\partial_n} & \cdots & \xrightarrow{\partial_1} & k[G]^{d_0} & \xrightarrow{\partial_0} & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
\cdots & \xrightarrow{\partial^{\mathbb{Z}}_{n+2}} & k[\mathbb{Z}]^{d_{n+1}} & \xrightarrow{\partial^{\mathbb{Z}}_{n+1}} & k[\mathbb{Z}]^{d_n} & \xrightarrow{\partial^{\mathbb{Z}}_n} & \cdots & \xrightarrow{\partial^{\mathbb{Z}}_1} & k[\mathbb{Z}]^{d_0} & \xrightarrow{\partial^{\mathbb{Z}}_0} & 0\,,
\end{array}$$

where the boundary maps are viewed as matrices and $\partial_p^{\mathbb{Z}}$ is obtained by applying the map $G \to \mathbb{Z}$ to each entry of the matrix ∂_p. Note that the homology of the bottom chain complex is $\mathrm{H}_\bullet(G;k[\mathbb{Z}])$.

To apply results on rank functions, we need the boundary maps to be between finitely generated free modules. However, d_{n+1} is not finite in general, so we must modify the chain complexes as follows. Since $k[\mathbb{Z}]$ is Noetherian and $k[\mathbb{Z}]^{d_n}$ is finitely generated, $\operatorname{im} \partial^{\mathbb{Z}}_{n+1}$ is a finitely generated submodule of $k[\mathbb{Z}]^{d_n}$. The preimage of a finite generating set of $\operatorname{im} \partial_{n+1}^{\mathbb{Z}}$ is contained in a finitely generated free summand F of $k[\mathbb{Z}]^{d_{n+1}}$. Notice that the homology of

$$F \to k[\mathbb{Z}]^{d_n} \to \cdots \to k[\mathbb{Z}]^{d_0} \to 0$$

is still $\mathrm{H}_i(G; k[\mathbb{Z}])$ for $i \leqslant n$. The preimage of F in $k[G]^{d_{n+1}}$ is again a finitely generated free summand $\widehat{F}$ of $k[G]^{d_{n+1}}$. It suffices to show that the homology of

$$\mathcal{D}_{k[G]} \otimes_{k[G]} \widehat{F} \to \mathcal{D}_{k[G]} \otimes_{k[G]} k[G]^{d_n} \to \cdots \to \mathcal{D}_{k[G]} \otimes_{k[G]} k[G]^{d_0} \to 0$$

vanishes in degrees at most n to show that $b_i^{\mathcal{D}_{k[G]}}(G) = 0$ for all $i \leqslant n$.

We assume that d_{n+1} is finite and that $F = k[\mathbb{Z}]^{d_{n+1}}$ and $\widehat{F} = k[G]^{d_{n+1}}$. Since, for every $i \leqslant n$, the homology $\mathrm{H}_i(G; k[\mathbb{Z}])$ is finite-dimensional as a k-vector space, it must be torsion as an $k[\mathbb{Z}]$-module. Therefore, $\operatorname{rk}_{\mathbb{Z}} \partial_{i+1}^{\mathbb{Z}} = \dim \ker \partial_i^{\mathbb{Z}}$ for every $i \leqslant n$, where $\operatorname{rk}_{\mathbb{Z}}(A)$ denotes the torsion-free rank of the image of a $k[\mathbb{Z}]$-matrix A. Now, for each $i \leqslant n$, we have short exact sequences

$$0 \to \ker \partial_i^{\mathbb{Z}} \to k[G]^{d_i} \to \operatorname{im} \partial_i^{\mathbb{Z}} \to 0$$

which implies that $d_i = \dim \ker \partial_i^{\mathbb{Z}} + \operatorname{rk}_{\mathbb{Z}} \partial_i^{\mathbb{Z}} = \operatorname{rk}_{\mathbb{Z}} \partial_{i+1}^{\mathbb{Z}} + \operatorname{rk}_{\mathbb{Z}} \partial_i^{\mathbb{Z}}$. Hence,

$$\begin{aligned} d_i - \operatorname{rk}_{\mathcal{D}_{k[G]}} \partial_i &= \dim_{\mathcal{D}_{k[G]}} \ker(\mathcal{D}_{k[G]}^{d_i} \xrightarrow{\partial_i} \mathcal{D}_{k[G]}^{d_{i-1}}) \\ &\geqslant \operatorname{rk}_{\mathcal{D}_{k[G]}} \partial_{i+1} \\ &\geqslant \operatorname{rk}_{\mathbb{Z}} \partial_{i+1}^{\mathbb{Z}} \\ &= d_i - \operatorname{rk}_{\mathbb{Z}} \partial_i^{\mathbb{Z}} \\ &\geqslant d_i - \operatorname{rk}_{\mathcal{D}_{k[G]}} \partial_i, \end{aligned}$$

where we have used the universality of $\mathcal{D}_{k[G]}$ (see Theorem 2.3.14). Thus,

$$\operatorname{rk}_{\mathcal{D}_{k[G]}} \partial_{i+1} = \dim_{\mathcal{D}_{k[G]}} \ker(\mathcal{D}_{k[G]}^{d_i} \xrightarrow{\partial_i} \mathcal{D}_{k[G]}^{d_{i-1}}),$$

and therefore $b_i^{\mathcal{D}_{k[G]}}(G) = 0$ for all $i \leqslant n$. □

Corollary 3.2.7. *Let G be a RFRS group of type* FP_n.

(1) *Suppose k and k' are division rings of the same characteristic. Then G virtually fibres with kernel of type* $\mathrm{FP}_n(k)$ *if and only if it virtually fibres with kernel of type* $\mathrm{FP}_n(k')$.

(2) *If G virtually fibres with kernel of type* $\mathrm{FP}_n(k)$ *for some division ring k, then it virtually fibres with kernel of type* $\mathrm{FP}_n(\mathbb{Q})$.

Proof. Let N be the kernel of a virtual of G to $\mathbb{Z}$. By the universal coefficient, $b_i(N;k) = b_i(N;k')$ and $b_i(N;k) \geqslant b_i(N;\mathbb{Q}) = b_i(N)$. The corollary then follows from Theorem 3.2.6. ☐

Note that Corollary 3.2.7 could also be obtained via approximation results such as [AOS24, Theorem 3.6].

3.2.2 Simultaneous fibring

In this subsection, we will show that if G is a RFRS group of finite type that admits a collection of virtual fibrings $\{\varphi_j\}_{j\in J}$ whose kernels are, respectively, of type $\mathrm{FP}_{n_j}(k_j)$ for some division rings k_j and integers n_j, then there is a single virtual algebraic fibration φ whose kernel is of type $\mathrm{FP}_{n_j}(k_j)$ for all $j \in J$ (see Theorem 3.2.10 for a stronger statement involving homotopical finiteness properties).

Let $\cdots \to C_1 \to C_0 \to 0$ be a chain complex of free R-modules for some ring R, with boundary maps $\partial_i \colon C_i \to C_{i-1}$. Then $\mathrm{H}_i(C_\bullet) = 0$ for all $i \leqslant n$ if and only if there are *chain contractions* $s_i \colon C_i \to C_{i+1}$ for a $i \leqslant n$, which by definition are R-module morphisms satisfying

$$\mathrm{id}_{C_i} = \partial_{i+1} \circ s_i + s_{i-1} \circ \partial_i.$$

Let $P \subset \mathbb{Z}$ be a collection of prime numbers, and let $P^{-1}\mathbb{Z}$ be the localisation of $\mathbb{Z}$ at the multiplicative set generated by P. The following proposition appears in joint work with Italiano and Kielak [FIK25, Proposition 3.6].

Proposition 3.2.8. *Let G be a group of type FP_n and let $\chi \colon G \to \mathbb{R}$ be a character. If $\mathrm{H}_i(G; \widehat{\mathbb{Q}[G]}^\chi) = 0$ for $i \leqslant n$, then there is a finite set of primes P such that*

$$\mathrm{H}_i(G; \widehat{P^{-1}\mathbb{Z}[G]}^\chi) = 0$$

for all $i \leqslant n$.

Proof. Let $C_\bullet \to \mathbb{Z} \to 0$ be a free resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$, where the modules C_i are finitely generated for each $i \leqslant n$. Fix bases for all of the free modules C_i; via this choice of basis, we will view the homomorphisms of free modules below as matrices with entries in an appropriate ring. By induction on i, we will construct chain contractions

$$s_i \colon \widehat{\mathbb{Q}[G]}^\chi \otimes_{\mathbb{Z}[G]} C_i \to \widehat{\mathbb{Q}[G]}^\chi \otimes_{\mathbb{Z}[G]} C_{i+1}$$

whose entries lie in $\widehat{P_i^{-1}\mathbb{Z}[G]}^\chi$ for some finite set of primes P_i, where we are viewing the maps s_i as matrices via the choice of bases for the modules C_i. The proposition follows by taking $P = P_n$. We will denote the boundary maps of the chain complex $\widehat{\mathbb{Q}[G]}^{\chi} \otimes_{\mathbb{Z}[G]} C_\bullet$ by

$$\partial_i \colon \widehat{\mathbb{Q}[G]}^{\chi} \otimes_{\mathbb{Z}[G]} C_i \to \widehat{\mathbb{Q}[G]}^{\chi} \otimes_{\mathbb{Z}[G]} C_{i-1}.$$

When viewed as matrices via the fixed bases, they have coefficients in $\mathbb{Z}[G]$.

The base case is $i = -1$, and s_{-1} is the zero map, so there is nothing to show. Assume $-1 < i \leqslant n$ and that we have constructed chain contractions $s_{-1}, \dots, s_{i-1}$ with entries lying in $\widehat{P_{i-1}^{-1}\mathbb{Z}[G]}^{\chi}$ for some finite set of primes P_{i-1}. By assumption, there is a chain contraction

$$\sigma_i \colon \widehat{\mathbb{Q}[G]}^{\chi} \otimes_{\mathbb{Z}[G]} C_i \to \widehat{\mathbb{Q}[G]}^{\chi} \otimes_{\mathbb{Z}[G]} C_{i+1}$$

such that

$$\mathrm{id}_{\widehat{\mathbb{Q}[G]}^{\chi} \otimes_{\mathbb{Z}[G]} C_i} = \partial_{i+1} \circ \sigma_i + s_{i-1} \circ \partial_i.$$

We can write $\sigma_i = \overline{\sigma}_i + \sigma_i'$, where $\overline{\sigma}_i$ has entries in $\mathbb{Q}[G]$ and σ_i' has entries in $\widehat{\mathbb{Q}[G]}^{\chi}$, and

$$\partial_{i+1} \circ \overline{\sigma}_i + s_{i-1} \circ \partial_i = \mathrm{id}_{\widehat{\mathbb{Q}[G]}^{\chi} \otimes_{\mathbb{Z}[G]} C_i} - A,$$

where $A \colon \widehat{\mathbb{Q}[G]}^{\chi} \otimes_{\mathbb{Z}[G]} C_i \to \widehat{\mathbb{Q}[G]}^{\chi} \otimes_{\mathbb{Z}[G]} C_i$ is a matrix over $\widehat{\mathbb{Q}[G]}^{\chi}$ whose entries all have positive support under χ. Since $\overline{\sigma}_i$ has finitely many nonzero entries, it follows that A has entries in $\widehat{P_i^{-1}\mathbb{Z}[G]}^{\chi}$ for some finite set of primes P_i. Note that $\mathrm{id}_{\widehat{\mathbb{Q}[G]}^{\chi} \otimes_{\mathbb{Z}[G]} C_i} - A$ is invertible with inverse $\sum_{j=0}^{\infty} A^j$, and moreover that

$$\partial_i \circ A = \partial_i \circ (\mathrm{id}_{\widehat{\mathbb{Q}[G]}^{\chi} \otimes_{\mathbb{Z}[G]} C_i} - \partial_{i+1} \circ \overline{\sigma}_i - s_{i-1} \circ \partial_i) = 0.$$

Then

$$\partial_{i+1} \circ \overline{\sigma}_i \circ (\mathrm{id}_{\widehat{\mathbb{Q}[G]}^{\chi} \otimes_{\mathbb{Z}[G]} C_i} - A)^{-1} + s_{i-1} \circ \partial_i = \mathrm{id}_{\widehat{\mathbb{Q}[G]}^{\chi} \otimes_{\mathbb{Z}[G]} C_i},$$

so we may take $s_i = \overline{\sigma}_i \circ (\mathrm{id}_{\widehat{\mathbb{Q}[G]}^{\chi} \otimes_{\mathbb{Z}[G]} C_i} - A)^{-1}$, which has entries in $\widehat{P_i^{-1}\mathbb{Z}[G]}^{\chi}$, since $(\mathrm{id}_{\widehat{\mathbb{Q}[G]}^{\chi} \otimes_{\mathbb{Z}[G]} C_i} - A)^{-1} = \sum_{j=0}^{\infty} A^j$. □

We pause to recall the definition of the homotopical sigma invariant, which will be used in the proof of Theorem 3.2.10. Given a group G, recall that the character sphere $S(G)$ is the set of equivalence classes of non-zero characters $\chi \colon G \to \mathbb{R}$, where the equivalence is given by scaling by a non-zero constant. Suppose that G is a group of type F_n for some $n \geqslant 0$, and let X be a $K(G,1)$ with compact n-skeleton. Let $\widetilde{X}$ denote the universal cover of X, and let $f_\chi \colon \widetilde{X} \to \mathbb{R}$ be a G-equivariant height map induced by χ. For each $a \in \mathbb{R}$, let $\widetilde{X}_a = f_\chi^{-1}([a, \infty[)$. The directed system of spaces $\widetilde{X}_a$ (under inclusion) is *essentially k-connected* if the directed limit $\varinjlim_a \pi_i(\widetilde{X}_a)$

is trivial for all $i \leqslant k$. The *nth homotopical sigma invariant* ${}^*\Sigma^n(G)$ is the subset of $S(G)$ defined as follows: $[\chi] \in^* \Sigma^n(G)$ if and only if a choice of directed system $\widetilde{X}_a$ as above is essentially $(n-1)$-connected (this will then imply all such choices are $(n-1)$-connected). We will use the following basic properties of the homotopical sigma invariants, which were established by Renz in his thesis where he introduced the concept.

Theorem 3.2.9 ([Ren88, Satz A and Satz C])**.** *Let G be a group of type* F_n*.*

(1) *${}^*\Sigma(G)$ is open in $S(G)$.*

(2) *If $\chi\colon G \to \mathbb{Z}$ is an epimorphism, then* $\ker\chi$ *is of type* F_n *if and only if* $[\pm\chi] \in^* \Sigma(G)$*.*

We are now ready to prove the simultaneous fibring theorem.

Theorem 3.2.10. *Let $\{k_j\}_{j\in J}$ be a set of division rings, and let G be a RFRS group of finite type. Suppose that G admits virtual algebraic fibrations φ and φ_j with kernels of type* F_m *and of type* $\mathrm{FP}_{n_j}(k_j)$ *for each $j \in J$, respectively. Then there exists a finite-index subgroup $H \leqslant G$ and an epimorphism $\chi\colon H \to \mathbb{Z}$ such that* $\ker(\chi)$ *is of type* F_m *and of type* $\mathrm{FP}_{n_j}(k_j)$ *for all $j \in J$.*

Proof. Let S be the set of non-negative integers p such that there exists at least one field k_j of characteristic p. Let $n = \mathrm{cd}(G)$; we may then assume that $m, n_j \leqslant n$ for all $j \in J$. For each $p \in S$, let n_p be the maximal integer n_j such that k_j is of characteristic p. We will prove that there exists a virtual algebraic fibration χ with kernel of type F_m and of type $\mathrm{FP}_{n_p}(\mathbb{F}_p)$, where $\mathbb{F}_0 := \mathbb{Q}$, which implies the result.

By passing to a finite-index subgroup if necessary, assume that $\varphi\colon G \to \mathbb{Z}$ is a fibration with kernel of type F_m. If $S = \varnothing$, then we are done. Otherwise, let $l = \max\{n_p : p \in S\}$. By Corollary 3.2.7, this implies that G admits a virtual fibration φ virtually fibres with with kernel of type $\mathrm{FP}_l(\mathbb{Q})$, and therefore $b_i^{(2)}(G) = 0$ for $i \leqslant l$. By Theorem 3.2.2, there is a finite-index subgroup $H \leqslant G$ and an antipodally symmetric open set $U \subseteq S(H)$ such that $\overline{U} \supseteq S(G)$ such that $\mathrm{H}_i(G; \widehat{\mathbb{Q}[G]}^{\psi}) = 0$ for all $\psi \in U$. The fact that $\ker\varphi|_H$ is of type F_m means that $[\varphi|_H]$ and $[-\varphi|_H]$ are contained in the mth *homotopical* sigma invariant ${}^*\Sigma(H)$, which is introduced in [Ren88]. The main feature of the invariant we will use is that it is open [Ren88, Satz A] in $S(H)$. This, combined with the fact that $\overline{U} \supseteq S(G) \ni \psi|_H$ implies that we can choose a new map $\varphi_0\colon H \to \mathbb{Z}$ whose kernel is of type F_m and of type $\mathrm{FP}_l(\mathbb{Q})$. By Proposition 3.2.8, there is a finite set of primes P such that $\ker(\varphi_0)$ is also of type $\mathrm{FP}_l(P^{-1}\mathbb{Z})$.

Consider the (finite) set $P \cap S = \{p_1, \dots, p_k\}$. We will prove by induction on k that there is an epimorphism $\varphi_k \colon H_k \to \mathbb{Z}$ such that $\ker(\varphi_k)$ is of type F_m, type $\mathrm{FP}_l(P^{-1}\mathbb{Z})$, and type $\mathrm{FP}_{n_{p_i}}(\mathbb{F}_{p_i})$ for each $i \in \{1, \dots, k\}$, and where $H_k \leqslant H$ is a subgroup of finite index. If $k = 0$ (i.e. if $P \cap S = \varnothing$), then there is nothing to show. Assume that we have proven the claim when $P \cap S$ contains $k-1$ primes. By Corollary 3.2.7, we know that G virtually fibres with kernel of type $\mathrm{FP}_{n_{p_k}}(\mathbb{F}_{p_k})$, and therefore $b_i^{(2)}(H_{k-1}; \mathbb{F}_{p_k}) = 0$ for all $i \leqslant n_{p_k}$. Thus, Theorem 3.2.2 gives the existence of a finite-index subgroup $H_k \leqslant H_{k-1}$ and an open set $U_k \subseteq S(H_k)$ such that $\overline{U_k} \supseteq S(H_{k-1})$ and $\mathrm{H}_i(H_k; \widehat{\mathbb{F}_{p_k}[H_k]}^{\psi}) = 0$ for all $\psi \in U_k$. By the inductive hypothesis, there is an algebraic fibration $\varphi_{k-1} \colon H_{k-1} \to \mathbb{Z}$ with kernel of type F_m, type $\mathrm{FP}_l(P^{-1}\mathbb{Z})$, and type $\mathrm{FP}_{n_{p_i}}(\mathbb{F}_{p_i})$ for each $i \in \{1, \dots, k-1\}$, and therefore $\varphi_{k-1}|_{H_k}$ lies in the open subset

$$^*\Sigma(H_k) \cap \bigcap_{i=1}^{k-1} \Sigma^{n_{p_i}}_{\mathbb{F}_{p_i}}(H_k; \mathbb{F}_{p_i}) \ \subseteq \ S(H_k).$$

Because $\overline{U} \supseteq S(H_{k-1})$, we may choose a character $\varphi_k \colon H_k \to \mathbb{Z}$ such with kernel of type F_m, type $\mathrm{FP}_l(P^{-1}\mathbb{Z})$, and type $\mathrm{FP}_{n_{p_i}}(\mathbb{F}_{p_i})$ for each $i \in \{1, \dots, k\}$.

There is a ring homomorphism $P^{-1}\mathbb{Z} \to \mathbb{F}_p$ for every $p \notin P$. Thus, $\ker(\varphi_k)$ is also of type $\mathrm{FP}_l(\mathbb{F}_p)$ for each $p \notin P$. Thus, Let $\chi = \varphi_k$. Then $\chi = \varphi_k$ is as desired. □

Corollary 3.2.11. *Let G be a RFRS group of type* FP. *The following are equivalent:*

(1) *There is a finite-index subgroup $H \leqslant G$ and an epimorphism $\varphi \colon H \to \mathbb{Z}$ whose kernel is of type* $\mathrm{FP}(k)$ *for all fields;*
(2) *G is k-L^2-acyclic for every field k.*

In [IMM24], Italiano–Martelli–Migliorini gave an example of a hyperbolic (cusped) 7-manifold M^7 whose fundamental group algebraically fibres with a finitely presented kernel. The manifold M^7 is constructed using reflections of a right-angled hyperbolic polytope, and therefore its fundamental group is a subgroup of a right-angled Coxeter group. It follows that $\pi_1(M^7)$ is virtually RFRS, and it is clearly of finite type. Since $\pi_1(M^7)$ is also a (non-uniform) lattice in $\mathrm{PO}(7,1)$, all of its L^2-Betti numbers vanish. Theorem 3.2.3 thus implies that $\pi_1(M^7)$ virtually fibres with kernel of type $\mathrm{FP}(\mathbb{Q})$. Combining these observations with Theorem 3.2.10, we obtain the following corollary.

Corollary 3.2.12. *There is a finite-volume cusped hyperbolic* 7*-manifold whose fundamental group virtually algebraically fibres with a finitely presented kernel of type* $\mathrm{FP}(\mathbb{Q})$.

Chapter 4

On the cohomological dimension of normal subgroups

The material in this chapter is taken from the articles [Fis24b] and [FK24], and from unpublished work with Sánchez-Peralta.

4.1 Virtually free-by-cyclic RFRS groups

In the first section of this chapter, we give a short proof the fact that finitely presented RFRS groups of cohomological dimension two are virtually free-by-cyclic if and only if their second L^2-Betti number vanishes. The author is very grateful to Andrei Jaikin-Zapirain for communicating a simplification of his original argument, without which the proof would be far less elegant. A generalisation of the following result will be given in Theorem 4.3.9.

Theorem 4.1.1. *Let G be a finitely presented RFRS group of cohomological dimension at most two. Then G is virtually free-by-cyclic if and only if $b_2^{(2)}(G) = 0$.*

Proof. A free-by-cyclic group has vanishing second L^2-Betti number by, for instance, [Gab02, Théorème 6.6] (see also Theorem 2.5.5), and therefore so do virtually free-by-cyclic groups. Thus, assume that $b_2^{(2)}(G) = 0$. By Proposition 2.5.4(iii), we have that $\mathrm{H}^2(G; \mathcal{D}_{\mathbb{Q}[G]}) = 0$, and thus the same argument as in the proof of [Kie20b, Theorem 5.2] yields a finite-index subgroup $H \leqslant G$ and an epimorphism $\chi \colon H \to \mathbb{Z}$ such that $\mathrm{H}^2(H; \widehat{\mathbb{Q}[H]}^{\pm\chi}) = 0$.

We will prove that $N := \ker(\chi)$ is free by proving that it is of cohomological dimension one and appealing to the Stallings–Swan Theorem [Sta68, Swa69]. Let M be an arbitrary right $\mathbb{Q}[N]$-module, let $t \in H$ map to a generator of $H/N \cong \mathbb{Z}$, and denote the coinduced module of M by $\sum_{i\in\mathbb{Z}} Mt^i$ (where we allow the sums to have infinite support). By Shapiro's Lemma, it will suffice to show that

$$\mathrm{H}^2(H;\sum_{i\in\mathbb{Z}} Mt^i)=0.$$

Consider the modules

$$M^\chi := \bigcup_{n\in\mathbb{Z}} \left\{ \sum_{i\geqslant n} m_i t^i \ : \ m_i \in M \right\} \quad \text{and} \quad M^{-\chi} := \bigcup_{n\in\mathbb{Z}} \left\{ \sum_{i\leqslant n} m_i t^i \ : \ m_i \in M \right\}$$

of $\sum_{i\in\mathbb{Z}} Mt^i$. Note that $M^{\pm\chi}$ is a $\widehat{\mathbb{Q}[H]}^{\pm\chi}$-module. By [Bro94, Proposition VIII.6.8] and the fact that H is finitely presented,

$$\mathrm{H}^2(H;M^{\pm\chi}) = \mathrm{H}^2(H;\widehat{\mathbb{Q}[H]}^{\pm\chi}) \otimes_{\widehat{\mathbb{Q}[H]}^{\pm\chi}} M^{\pm\chi} = 0.$$

The long exact sequence in the cohomology of H associated to the short exact sequence of coefficients

$$0 \to M \otimes_{\mathbb{Q}[N]} \mathbb{Q}[G] \to M^\chi \oplus M^{-\chi} \to \sum_{i\in\mathbb{Z}} Mt^i \to 0$$

contains the portion

$$0 = \mathrm{H}^2(H;M^\chi) \oplus \mathrm{H}^2(H;M^{-\chi}) \to \mathrm{H}^2(H;\sum_{i\in\mathbb{Z}} Mt^i) \to \mathrm{H}^3(H;M\otimes_{\mathbb{Q}[N]}\mathbb{Q}[H]) = 0,$$

and therefore $\mathrm{H}^2(H;\sum_{i\in\mathbb{Z}} Mt^i)=0$, as desired. □

Remark 4.1.2. As we will see below, the assumption in Theorem 4.1.1 that G be finitely presented is not necessary; it is sufficient to assume that G be finitely generated. This is because if G is of cohomological dimension two and its second L^2-Betti number vanishes, then G is homologically coherent by [JZL23, Theorem 3.10]. In particular, if G is finitely generated, then it is of type FP_2, and this is all that was actually needed in the proof.

4.2 The invariant Malcev–Neumann rings of Okun and Schreve

We now move away from RFRS groups and study residually poly-$\mathbb{Z}$ groups (recall that a finitely generated group is RFRS if and only if it is residually (poly-$\mathbb{Z}$ and virtually Abelian) by Theorem 2.4.2), with the aim of proving a generalisation of Theorem 4.1.1

in this class. Before doing this, we must recall a construction of Okun and Schreve from [OS24], where they give a simplification of Kielak's original description of the Linnell division ring of a RFRS group in terms of Novikov rings.

In this section and in what follows, an *order* $<$ on a group is a total bi-invariant ordering (these are often called bi-orders). If an order is one-sided, then we will specify it.

Consider the following general set up. Let Γ be a group, let k be a field such that $\mathcal{D}_{k[\Gamma]}$ exists, and suppose that $N \leqslant G$ is a pair of normal subgroups of Γ such that the quotient G/N is orderable and amenable. For each order $<$ on G/N, there is a representation

$$\iota_< \colon \mathcal{D}_{k[G]} \hookrightarrow \mathcal{D}_{k[N]} *_< G/N.$$

This is because the division closure of $k[G]$ in $\mathcal{D}_{k[N]} *_< G/N$ is Hughes-free. If $R \subseteq \mathcal{D}_{k[N]}$ is a Γ-conjugation invariant subring, then the corresponding *invariant Malcev–Neumann ring* is the subring of $\mathcal{D}_{k[G]}$ consisting of the elements whose representations under $\iota_<$ lie in $R *_< G/N$. More precisely, the invariant Malcev–Neumann ring is denoted and defined by

$$R ** G/N \ := \ \bigcap_< \iota_<^{-1}(R *_< G/N),$$

where the intersection in $\mathcal{D}_{k[G]}$ is taken over all orders on G/N.

Now suppose that G is a group with a residual normal chain $G = G_0 \geqslant G_1 \geqslant \dots$ such that each successive quotient G_i/G_{i+1} is orderable and amenable. Since orderable groups are locally indicable, G is residually (locally indicable and amenable), and therefore $\mathcal{D}_{k[G]}$ exists and is universal by Theorem 2.3.14. For each $j \geqslant 0$, define

$$R_j^j = k[G_j] \quad \text{and} \quad R_j^i = R_j^{i+1} ** G_i/G_{i+1} \quad \text{for } 0 \leqslant i < j$$

by reverse recursion on i. The rings R_i^j are the *inductive rings* associated to the residual chain $(G_i)_{i\geqslant 0}$. The main result of Okun–Schreve is the following, which gives a description of $\mathcal{D}_{k[G]}$ in terms of the inductive rings of $(G_i)_{i\geqslant 0}$.

Theorem 4.2.1 ([OS24, Theorem 5.1]). *With the above notation,* $\mathcal{D}_{k[G]} = \bigcup_{j\geqslant 0} R_j^0$.

As a consequence of this description, we obtain the following result that will be used below.

Proposition 4.2.2. *Let G be a residually (locally indicable and amenable) group, let k be a field, let $G = G_0 \geqslant G_1 \geqslant \dots$ be a normal residual chain such that each* *quotient G_i/G_{i+1} is orderable and amenable, and denote by R^i_j the inductive rings corresponding to this chain. Let $P_\bullet$ be a chain complex of projective $k[G]$-modules such that P_n is finitely generated for some $n \in \mathbb{Z}$. If $\mathrm{H}^n(P_\bullet; \mathcal{D}_{k[G]}) = 0$, then $\mathrm{H}^n(P_\bullet; R^0_j) = 0$ for some sufficiently large integer j.*

Proof. We may assume that $P_\bullet$ is a chain complex of free $k[G]$-modules such that P_n is finitely generated. Indeed, let Q_n be a projective module such that $P_n \oplus Q_n$ is finitely generated and free. The cohomology of the chain complex

$$\cdots \to P_{n+2} \to P_{n+1} \oplus Q_{n+1} \to P_n \oplus Q_n \to P_{n-1} \to \cdots$$

(with any coefficients) is isomorphic to that of $P_\bullet$. By iteratively taking free complements of the other projective modules in the chain complex, we may assume that they are all free, and that P_n is finitely generated.

We fix some notations and terminology that will be used in the rest of the proof. If M is a $k[G]$-module, then we denote $\mathrm{Hom}_{k[G]}(M, \mathcal{D}_{k[G]})$ by M^*. For each free $k[G]$-module P_i, fix an isomorphism $P_i \cong \bigoplus_{J_i} k[G]$. We say that a submodule $F_i \leqslant P_i$ is a *direct summand* if it corresponds to $\bigoplus_{J_i'} k[G]$ for some $J_i' \subseteq J_i$. Note that this is more restrictive than simply being a factor in a direct factor in an abstract direct sum decomposition of P_i.

We now focus on the portion $P_{n+1} \to P_n \to P_{n-1}$ of the chain complex. We want to choose finitely generated direct summands $F_{n\pm 1} \leqslant P_{n\pm 1}$ so that the cohomology of $F^*_{n+1} \leftarrow P^*_n \leftarrow F^*_{n-1}$ coincides with that of $P^*_{n+1} \leftarrow P^*_n \leftarrow P^*_{n-1}$. Since P_n is finitely generated, its image in P_{n+1} lies in a finitely generated direct summand $F_{n-1} \leqslant P_{n+1}$. Then the images of the maps $P^*_{n-1} \to P^*_n$ and $F^*_{n-1} \to P^*_n$ coincide, since every map on F_{n-1} extends to one on P_{n-1}. Note that this did not have anything to do with the choice of coefficients in $\mathcal{D}_{k[G]}$.

Since P_{n+1} is the directed union of its finitely generated direct summands, it follows that

$$\ker(P^*_n \to P^*_{n+1}) = \bigcap_{\substack{F_{n+1} \leqslant P_{n+1} \\ \text{f.g. direct summand}}} \ker(P^*_n \to F^*_{n+1}).$$

Because $\mathcal{D}_{k[G]}$ is a division ring, each $\ker(P^*_n \to F^*_{n+1})$ is a finite-dimensional $\mathcal{D}_{k[G]}$-module and every chain of such modules must have a minimal element (under inclusion). By Zorn's Lemma, there is a minimal such $\mathcal{D}_{k[G]}$-module, and therefore

$$\ker(P^*_n \to P^*_{n+1}) = \ker(P^*_n \to F^*_{n+1})$$

for some finitely generated direct summand $F_{n+1} \leqslant P_{n+1}$. Let $F_n = P_n$. Then the degree n cohomology of of $F^*_{n+1} \leftarrow F^*_n \leftarrow F^*_{n-1}$ coincides with that of $P^*_{n+1} \leftarrow P^*_n \leftarrow P^*_{n-1}$, as desired.

The coboundary maps of $F^*_{n+1} \leftarrow F^*_n \leftarrow F^*_{n-1}$ are maps between finitely generated modules over $\mathcal{D}_{k[G]}$, which we identify with finite matrices (note that these matrices will have entries in $k[G]$, since they are induced by maps of free $k[G]$-modules). Since $\mathcal{D}_{k[G]}$ is a division ring, for $i = n-1, n, n+1$ there are invertible matrices

$$M_i \colon \operatorname{Hom}_{k[G]}(F_i, \mathcal{D}_{k[G]}) \to \operatorname{Hom}_{k[G]}(F_i, \mathcal{D}_{k[G]})$$

that put the coboundary maps into Smith normal form. By Theorem 4.2.1, there is some $j \geqslant 0$ such that every entry of each matrix $M_i^{\pm 1}$ has coefficients in R_j^0. Thus, the matrices $M_i^{\pm 1}$ put the cochain complex

$$\operatorname{Hom}_{k[G]}(F_{n+1}, R_j^0) \leftarrow \operatorname{Hom}_{k[G]}(F_n, R_j^0) \leftarrow \operatorname{Hom}_{k[G]}(F_{n-1}, R_j^0)$$

into the same Smith normal form, and therefore its degree n cohomology vanishes.

To conclude, note that $\operatorname{Hom}_{k[G]}(F_{n-1}, R_j^0)$ and $\operatorname{Hom}_{k[G]}(P_{i-1}, R_j^0)$ have the same image in $\operatorname{Hom}_{k[G]}(P_n, R_j^0)$. The kernel of

$$\operatorname{Hom}_{k[G]}(P_n, R_j^0) \to \operatorname{Hom}_{k[G]}(P_{n+1}, R_j^0)$$

is contained in that of

$$\operatorname{Hom}_{k[G]}(P_n, R_j^-) \to \operatorname{Hom}_{k[G]}(F_{n+1}, R_j^0),$$

so we conclude that $\mathrm{H}^n(P_\bullet; R_j^0) \leqslant \mathrm{H}^n(F_\bullet; R_j^0) = 0$, as desired. □

4.3 Dimension drop in residual chains

4.3.1 Dimension of normal subgroups

We give an explicit definition of the weakest finiteness condition to which our arguments will apply. We will require groups to be of finite cohomological dimension, and to admit a projective resolution which is finitely generated in the top dimension.

Definition 4.3.1. Let G be a group and R be a ring. We say that G is of *type* $\mathrm{FTP}_n(R)$ if the trivial $R[G]$-module R admits a projective resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to R \to 0$$

where P_n is finitely generated.

Note that if G is of type $\mathrm{FP}(R)$, then G is of type $\mathrm{FTP}_{\mathrm{cd}_R(G)}(R)$, and if G is of type $\mathrm{FTP}_n(R)$, then $\mathrm{cd}_R(G) \leqslant n$.

Let $N \leqslant G$ be a pair of groups, and let M be a right $R[N]$-module for some ring R. The *coinduced module* is

$$\mathrm{Coind}_N^G(M) = \mathrm{Hom}_{R[N]}(R[G], M) \cong \sum_{t\in T} Mt$$

where T is a right transversal for N in G. By an abuse of notation, we will often denote the coinduced module by $\sum_{t\in N\backslash G} Mt$, where the choice of transversal is implicit.

Lemma 4.3.2. *Let R be a ring and let $1 \to N \to G \to Q \to 1$ be a short exact sequence of groups such that G is of type $\mathrm{FTP}_n(R)$, and let M be an $R[N]$-module. If there exist collections of $R[G]$-rings $\{S_i\}_{i\in I}$, S_i-modules $\{L_i\}_{i\in I}$, and $R[G]$-module homomorphisms $\{L_i \to \sum_{t\in Q} Mt\}_{i\in I}$ such that the induced map*

$$\bigoplus_{i\in I} L_i \to \sum_{t\in Q} Mt$$

is surjective and $\mathrm{H}^n(G; S_i) = 0$ for all $i \in I$, then $\mathrm{H}^n(N; M) = 0$.

Proof. Fix a projective resolution $0 \to P_n \to \cdots \to P_0 \to R \to 0$ of the trivial $R[G]$-module R such that P_n is finitely generated. We will prove that $\mathrm{H}^n(N, M) = 0$ by a series of reductions.

Claim 4.3.3. *It is suffices to show that $\mathrm{H}^n(G; \sum_{t\in Q} Mt) = 0$.*

Proof. This is immediate by Shapiro's Lemma. ◇

Claim 4.3.4. *It is suffices to show that $\mathrm{H}^n(G; \bigoplus_{i\in I} L_i) = 0$.*

Proof. Let K be the kernel of the surjection $\bigoplus_{i\in I} L_i \to \sum_{t\in Q} Mt$. Then the long exact sequence in cohomology associated to the surjection contains the portion

$$\mathrm{H}^n(G; \bigoplus_{i\in I} L_i) \to \mathrm{H}^n(G; \sum_{t\in Q} Mt) \to \mathrm{H}^{n+1}(G; K),$$

where K is the kernel of the surjection. But $\mathrm{cd}_R(G) \leqslant n$, so $\mathrm{H}^n(G; \bigoplus_{i\in I} L_i)$ surjects onto $\mathrm{H}^n(G; \sum_{t\in Q} Mt)$, which proves the claim. ◇

Claim 4.3.5. *It suffices to show that $\mathrm{H}^n(G; L_i) = 0$ for all $i \in I$.*

Proof. Because P_n is finitely generated, there is a natural isomorphism

$$\bigoplus_{i\in I} \mathrm{Hom}_{R[G]}(P_n, L_i) \cong \mathrm{Hom}_{R[G]}(P_n, \bigoplus_{i\in I} L_i).$$

Thus, there is a commutative diagram

$$\begin{array}{ccccc} \bigoplus_{i\in I} \mathrm{Hom}_{R[G]}(P_n; L_i) & \longrightarrow & \bigoplus_{i\in I} \mathrm{H}^n(G; L_i) & \longrightarrow & 0 \\ \cong\downarrow & & \downarrow & & \\ \mathrm{Hom}_{R[G]}(P_n; \bigoplus_{i\in I} L_i) & \longrightarrow & \mathrm{H}^n(G; \bigoplus_{i\in I} L_i) & \longrightarrow & 0 \end{array}$$

with exact rows. This immediately implies that the rightmost vertical map is an epimorphism, and thus proves the claim. ◇

Claim 4.3.6. *The natural map* $\mathrm{H}^n(G; S_i) \otimes_{S_i} L_i \to \mathrm{H}^n(G; L_i)$ *is surjective for all* $i \in I$.

Proof. Because P_n is finitely generated, there are natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{R[G]}(P_n, L_i) &\cong \mathrm{Hom}_{R[G]}(P_n, R[G]) \otimes_{R[G]} L_i \\ &\cong \mathrm{Hom}_{R[G]}(P_n, R[G]) \otimes_{R[G]} S_i \otimes_{S_i} L_i \\ &\cong \mathrm{Hom}_{R[G]}(P_n, S_i) \otimes_{S_i} L_i. \end{aligned}$$

and because tensoring is right exact, the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{R[G]}(P_n, S_i) \otimes_{S_i} L_i & \longrightarrow & \mathrm{H}^n(G; S_i) \otimes_{S_i} L_i & \longrightarrow & 0 \\ \cong\downarrow & & \downarrow & & \\ \mathrm{Hom}_{R[G]}(P_n; L_i) & \longrightarrow & \mathrm{H}^n(G; L_i) & \longrightarrow & 0 \end{array}$$

has exact rows. Therefore the rightmost vertical map is surjective, which proves the claim. ◇

The conclusion $\mathrm{H}^n(N, M) = 0$ then follows immediately from the claims, since we assume $\mathrm{H}^n(R; S_i) = 0$ for all $i \in I$. □

4.3.2 Residually poly-$\mathbb{Z}$ groups

A group G is *residually poly-*$\mathbb{Z}$ if it has a residual normal chain $G = G_0 \geqslant G_1 \geqslant \dots$ such that each quotient G/G_i is poly-$\mathbb{Z}$. If G is countable, then this is equivalent to the usual definition of a residual property, namely that every element $g \in G \smallsetminus \{1\}$ survives in a poly-$\mathbb{Z}$ quotient of G.

Lemma 4.3.7. *If G is a residually poly-$\mathbb{Z}$ group, then it admits a residual normal chain $G = G_0 \geqslant G_1 \geqslant \dots$ such that each consecutive quotient G_i/G_{i+1} is free Abelian of finite rank.*

Proof. Let $G = H_0 \geqslant H_1 \geqslant \dots$ be a residual normal chain such that each quotient G/H_i is poly-$\mathbb{Z}$. We will refine $(H_i)_{i\geqslant 0}$ so that it has free-Abelian consecutive quotients.

Fix some i and consider the quotient H_i/H_{i+1}, which is poly-$\mathbb{Z}$. Therefore there is a characteristic series

$$1 = Q_n \leqslant Q_{n-1} \leqslant \dots \leqslant Q_1 \leqslant Q_0 = H_i/H_{i+1}$$

such that each consecutive quotient is free-Abelian. Indeed, one inductively defines $Q_0 = H_i/H_{i+1}$ and $Q_{j+1} = \ker(Q_j \to \mathrm{H}_1(Q_j; \mathbb{Q}))$. Lifting the groups Q_i to G, one obtains a refinement of $(H_i)_{i\geqslant 0}$ whose successive quotients are free Abelian and such that all the subgroups are normal in G (this follows from the fact that the groups Q_j are characteristic in H_i/H_{i+1}). □

Lemma 4.3.7 shows that residually poly-$\mathbb{Z}$ chains admit normal residual chains for which we can define inductive rings as in Section 4.2. For the remainder of this subsection, fix a residual normal chain $G = G_0 \geqslant G_1 \geqslant \dots$ where the consecutive quotients G_i/G_{i+1} are free Abelian, and let R^i_j denote the inductive rings corresponding to this residual normal chain.

Fix an integer n and consider the poly-$\mathbb{Z}$ quotient G/G_n. It has a subnormal series $1 \leqslant G_{n-1}/G_n \leqslant \dots \leqslant G_0/G_n = G/G_n$ with free-Abelian successive quotients. For each choice of order on each G_i/G_{i+1}, there is an induced lexicographic right-invariant order on G/G_n (if the subnormal series were central, then the lexicographic order would be bi-invariant). Let M be an arbitrary $k[G_n]$-module, and consider the coinduced module

$$\mathrm{Coind}^G_{G_n}(M) \cong \sum_{t\in T} Mt \cong \sum_{t_0\in T_0} \cdots \sum_{t_{n-1}\in T_{n-1}} Mt_{n-1}\dots t_0$$

where T_i is a transversal for G_{i+1} in G_i and $T = T_{n-1}\cdots T_0$ is a right transversal for G_n in G. If $<$ is a right-invariant order on G/G_n, then we denote by $M *_< G/G_n$ the right $k[G]$-module consisting of the elements $(m_t t)$ such that $\{t : m_t \neq 0\}$ is well-ordered under $<$.

Lemma 4.3.8. *If $<_i$ is an an ordering on G_i/G_{i+1} for each $i \in \{0, \dots, n-1\}$ and $<$ is the induced lexicographic right-invariant order on G/G_n, then $M *_< G/G_n$ has a natural right R^0_n-module structure extending the $k[G]$-action.*

Proof. We will prove that $M *_< G_i/G_n$ is an R^i_n-module by reverse induction on i. The base case is when $i = n$, in which case there is nothing to show, since $R^n_n = k[G_n]$ and $M *_< G_n/G_n = M$.

Suppose that $M *_< G_{i+1}/G_n$ has a right R^{i+1}_n-module structure. By the definition of the lexicographic order, $M *_< G_i/G_n$ coincides with the submodule

$$(M *_< G_{i+1}/G_n) *_{<_i} G_i/G_{i+1} \subseteq \sum_{s \in T_i} (M *_< G_{i+1}/G_n)s.$$

The action of R^i_n can now be defined as follows. Let

$$\alpha \in R^i_n := R^{i+1}_n ** G_i/G_{i+1} \quad \text{and} \quad x = \sum_{s \in T_i} m_s s \in M_{<_i}\langle G_i/G_n\rangle,$$

and let $\sum_{t \in G_i/G_{i+1}} r_t t$ be the image of α in $R^{i+1}_n *_{<_i} G_i/G_{i+1}$ under the representation $\iota_{<_i}$. By an abuse of notation, we now identify α with this image, and thus speak of the support of α. If $s, t \in T_i$, let $g_{s,t} \in G_{i+1}$ and $u_{s,t} \in T_i$ be the unique elements such that $st = g_{s,t}u_{s,t}$. Put

$$x \cdot \alpha \ := \sum_{s,t \in T_i} \left((m_s \cdot s r_t s^{-1}) g_{s,t}\right) u_{s,t}.$$

This is well defined, because $srs^{-1} \in R^{i+1}_n$ as R^{i+1}_n is G-conjugation invariant, and therefore the action $m_s \cdot s r_t s^{-1}$ is defined by our inductive hypothesis, and thus $(m_s \cdot s r_t s^{-1}) g_{s,t}$ is defined since $M_<\langle G_{i+1}/G_n\rangle$ is a right $k[G_{i+1}]$-module. Moreover, since x and α both have well-ordered supports with respect to the *bi*order $<_i$, we have that $\operatorname{supp}(x)\operatorname{supp}(\alpha)$ has well-ordered image in G_i/G_{i+1} and moreover that there are only finitely many pairs $(s,t) \in \operatorname{supp}(x) \times \operatorname{supp}(\alpha)$ such that $stG_{i+1} = g$ for any given $g \in G_i/G_{i+1}$. It is now not difficult to check that this defines an R^i_n-module action extending the $k[G_i]$-action. □

We are now ready to prove the main result of this chapter, which is an extension of Theorem 3.2.3.

Theorem 4.3.9. *Let G be a residually poly-$\mathbb{Z}$ group of type $\mathrm{FTP}_n(k)$ for some division ring k. The following are equivalent:*

(1) $b^{(2)}_n(G; k) = 0$;

(2) *If $G = G_0 \geqslant G_1 \geqslant \dots$ is any normal residual chain such that G/G_i is poly-$\mathbb{Z}$ for each i, then $\mathrm{cd}_k(G_i) < n$ for sufficiently large i.*

Proof. Suppose that $b_n^{(2)}(G;k) = 0$ and fix a residual normal chain $G = G_0 \geqslant G_1 \geqslant \dots$ such that each quotient G/G_i is poly-$\mathbb{Z}$. By Lemma 4.3.7 and by possibly refining the residual chain, we may also assume that the consecutive quotients G_i/G_{i+1} are finitely generated free Abelian.

By Proposition 4.2.2, there is some integer i for which $\mathrm{H}^n(G; R_i^0) = 0$. We claim that $\mathrm{cd}_k(G_i) < n$. Let M be an arbitrary $k[G_i]$-module; we will prove that $\mathrm{H}^n(G_i; M) = 0$. Consider the normal series

$$1 \leqslant G_{i-1}/G_i \leqslant \cdots \leqslant G_1/G_i \leqslant G_0/G_i = G/G_i$$

whose consecutive quotients are finitely generated free Abelian. In each consecutive quotient G_j/G_{j+1}, choose a preferred basis (as a free Abelian group). If G_j/G_{j+1} has a rank d_j, then this choice determines 2^{d_j} lexicographic orders on G_j/G_{j+1}. Hence, if G/G_i has Hirsch length h, then we have identified $N := 2^h$ lexicographic right-invariant orders on G/G_i, which we denote by $<_1, \dots, <_N$.

For each $l \in \{1, \dots, N\}$, there is an inclusion $M *_{<_l} G/G_i \hookrightarrow \sum_{t \in G/G_i} Mt$, and these inclusions induce a surjection

$$\bigoplus_{l=1}^{N} M *_{<_l} G/G_i \to \sum_{t \in G/G_i} Mt.$$

By Lemma 4.3.8, each $k[G_i]$-module $M *_{<_l} G/G_i$ carries a natural R_i^0-module structure. Thus, all the conditions of Lemma 4.3.2 are met, and so we conclude that $\mathrm{H}^n(G_i; M) = 0$, as desired.

Conversely, if $\mathrm{cd}_k(G_i) < n$ for some i, then $b_n^{(2)}(G_i;k) = 0$. Since the quotient G/G_i is poly-$\mathbb{Z}$, we conclude that $b_n^{(2)}(G;k) = 0$ by Theorem 2.5.5. □

We now specialise the result to dimension 2, where we obtain some of the strongest consequences.

Corollary 4.3.10. *Let G be a finitely generated residually poly-$\mathbb{Z}$ group satisfying $\mathrm{cd}_k(G) \leqslant 2$ for some field k. Then $b_2^{(2)}(G;k) = 0$ if and only if G is free-by-(poly-$\mathbb{Z}$).*

Proof. Suppose that $b_2^{(2)}(G;k) = 0$. By [JZL23, Theorem 3.10], G is homologically coherent and therefore is of type $\mathrm{FP}_2(k)$. In particular, G is of type $\mathrm{FTP}_2(k)$, and hence the claim follows from Theorem 4.3.9 and the Stallings–Swan Theorem [Sta68, Swa69]. The converse is again an immediate consequence of Theorem 2.5.5. □

There is an increasing number of results relating the vanishing of the second L^2-Betti number with coherence of groups of cohomological dimension 2 (see, e.g., [KKW22, KL24, JZL23, Wis20a]). In fact, Wise conjectures that the two phenomena should be equivalent (see [Wis22b, Conjecture 2.6] and [Wis20a, Problems 9 and 11]). We offer more evidence for one direction of this conjecture below. We thank Marco Linton for communicating an argument that free-by-(poly-$\mathbb{Z}$) groups of cohomological dimension 2 are coherent. If the reader is only interested in RFRS groups, then they can appeal to Theorem 4.1.1 and the result of Feighn–Handel stating that free-by-cyclic groups are coherent [FH99] instead. Recall that a ring is *left (resp. right) coherent* if all of its finitely generated left (resp. right) ideals are finitely presented. A group algebra is left coherent if and only if it is right coherent, so we drop the left/right specification.

Corollary 4.3.11. *Let G be a residually poly-$\mathbb{Z}$ group satisfying $\operatorname{cd}_k(G) \leqslant 2$ for some field k. If $b_2^{(2)}(G;k) = 0$, then G and $k[G]$ are coherent.*

Proof. It suffices to assume that G is finitely generated, since $b_2^{(2)}(H;k) = 0$ for all subgroups of $H \leqslant G$ by [FM23, Lemma 3.21], and a group G (resp. group algebra $k[G]$) is coherent if and only if H (resp. $k[H]$) is coherent for all finitely generated subgroups $H \leqslant G$. Hence, we can assume that G is free-by-(poly-$\mathbb{Z}$). We will need the following claim, whose proof is essentially identical to that of [JZL23, Theorem 3.4].

Claim 4.3.12. *Let G be a free-by-amenable group. If $\mathcal{D}_{k[G]}$ exists, then it is of weak dimension at most one as a $k[G]$-module, meaning that $\operatorname{Tor}_2^{k[G]}(M, \mathcal{D}_{k[G]})$ for all right $k[G]$-modules M.*

Proof. By [Tam54], we have $\mathcal{D}_{k[G]} \cong \operatorname{Ore}(\mathcal{D}_{k[F]} * G/F)$, where F is a free normal subgroup of G such that G/F is amenable. Since Ore localisation is a flat functor, it suffices to prove that $\mathcal{D}_{k[F]} * G/F$ is of weak dimension at most one as a $k[G]$-module. Let M be an arbitrary right $k[G]$-module. By Shapiro's Lemma, we have

$$\operatorname{Tor}_2^{k[G]}(M, \mathcal{D}_{k[F]} * G/F) \cong \operatorname{Tor}_2^{k[F]}(M, \mathcal{D}_{k[F]}) = 0$$

since $\operatorname{cd}_k(F) \leqslant 1$. $\diamond$

In particular, $\mathcal{D}_{k[G]}$ is of weak dimension at most one as a $k[G]$-module, since G is free-by-(poly-$\mathbb{Z}$). Hence, $k[G]$ is coherent by [JZL23, Corollary 3.2].

Finally, we show that G is a coherent group. As G is free-by-(poly-$\mathbb{Z}$), we prove coherence of G by induction on the Hirsch length of the poly-$\mathbb{Z}$ quotient. If the

Hirsch length is 0, then G is free and therefore coherent. Now suppose that the Hirsch length is $n > 0$. Then G splits as an ascending HNN extension of a group H, which is free-by-(poly-$\mathbb{Z}$ of Hirsch length $n-1$). By induction, H is coherent. By [JZL23, Theorem 1.3], it suffices to show that G is homologically coherent, i.e. that every finitely generated subgroup of G is of type $\mathrm{FP}_2(k)$. But this follows from the fact that $k[G]$ is coherent. □

4.3.3 Vanishing second homology, parafree groups, and a question of Wise

In [Wis20b], Wise proved that if G is a finitely presented RFRS group, then $b_2^{(2)}(G) \leqslant b_2(G)$. Jaikin-Zapirain subsequently showed that this is a special case of universality; more precisely, he showed that if G is a group of type $\mathrm{FP}_2(\mathbb{Q})$ such that the Linnell division ring is universal, then $b_2^{(2)}(G) \leqslant b(G)$. We begin by proving that the assumption that G be of type $\mathrm{FP}_2(\mathbb{Q})$ is not necessary. In particular, this result applies to RFRS groups by Theorem 2.3.14. We refer the reader to the background on specialisations in Section 2.3.4, which will be used in the proof of the following proposition.

Lemma 4.3.13. *Let G be a finitely generated group, let k be a field, and let $\mathcal{D}_1$ and $\mathcal{D}_2$ be $k[G]$-division rings such that there is a specialisation $\rho\colon \mathcal{D}_1 \to \mathcal{D}_2$. If $\mathrm{cd}_k(G) \leqslant 2$, then $b_2(G;\mathcal{D}_1) \leqslant b_2(G;\mathcal{D}_2)$.*

Proof. There is nothing to show if $b_2(G;\mathcal{D}_2) = \infty$, so we assume $b_2(G;\mathcal{D}_2) < \infty$. Let $0 \to P_2 \to P_1 \to P_0 \to k \to 0$ be a projective resolution of the trivial $k[G]$-module k, where P_0 and P_1 are finitely generated free $k[G]$-modules. Let $D \subseteq \mathcal{D}_1$ be the domain of ρ. Since projective modules over local rings are free by Kaplansky's Theorem [Kap58], there is some cardinal α such that $D \otimes_{k[G]} P_2 \cong D^{\oplus\alpha}$ as left D-modules, and therefore

$$\mathcal{D}_i \otimes_{k[G]} P_2 \cong \mathcal{D}_i \otimes_D D \otimes_{k[G]} \otimes P_2 \cong \mathcal{D}_i^{\oplus\alpha}$$

for $i = 1, 2$. The assumptions that $b_2(G;\mathcal{D}_2) < \infty$ and that P_1 is finitely generated imply that $\alpha < \infty$.

Since $\alpha < \infty$, there is a finitely generated free $k[G]$-module F and a homomorphism $\partial\colon F \to P_2$ such that

$$\mathcal{D}_2 \otimes_{k[G]} F \to \mathcal{D}_2 \otimes_{k[G]} P_2 \cong \mathcal{D}_2^{\alpha}$$

is an epimorphism. By passing to a free submodule of F, we may assume that the above map is an isomorphism. For $i = 1, 2$, the maps $\mathcal{D}_i \otimes_{k[G]} F \to \mathcal{D}_i \otimes_{k[G]} P_2$

are obtained by applying $\mathcal{D}_i \otimes_D -$ to the map of finitely generated free D-modules $D \otimes_{k[G]} F \to D \otimes_{k[G]} P_2$. But ρ is also a specialisation of D-division rings, so the rank functions still satisfy $\operatorname{rk}_{\mathcal{D}_1} \geqslant \operatorname{rk}_{\mathcal{D}_2}$ when viewed as rank functions on D by Theorem 2.3.13. It follows that

$$\mathcal{D}_1 \otimes_{k[G]} F \to \mathcal{D}_1 \otimes_{k[G]} P_2 \cong \mathcal{D}_1^\alpha$$

is also an isomorphism, and therefore that the chain complexes

$$0 \to \mathcal{D}_i \otimes_{k[G]} F \to \mathcal{D}_i \otimes_{k[G]} P_1 \to \mathcal{D}_i \otimes_{k[G]} P_0 \to 0$$

compute the Betti numbers $b_n(G; \mathcal{D}_i)$ for both $i = 1, 2$. Once again using the fact that $\operatorname{rk}_{\mathcal{D}_1} \geqslant \operatorname{rk}_{\mathcal{D}_2}$, we have

$$b_2(G; \mathcal{D}_1) = \alpha - \operatorname{rk}_{\mathcal{D}_1} \partial \leqslant \alpha - \operatorname{rk}_{\mathcal{D}_2} \partial = b_2(G; \mathcal{D}_2). \qquad \square$$

As a corollary, we solve a problem of Wise [Wis20b, Problem 6.5].

Corollary 4.3.14. *Let X be an aspherical 2-complex such that $\pi_1(X)$ is finitely generated and RFRS. If $b_2(X) = 0$, then $\pi_1(X)$ is virtually free-by-cyclic.*

Proof. By Theorem 2.3.14, the Linnell division ring $\mathcal{D}_{\mathbb{Q}[\pi_1(X)]}$ is universal for $\mathbb{Q}[\pi_1(X)]$, and therefore there is a specialisation $\mathcal{D}_{\mathbb{Q}[\pi_1(X)]} \to \mathbb{Q}$. $\square$

Our next application concerns parafree groups. A group G is *parafree* if G is residually nilpotent and there is a free group F such that G and F have the same set of isomorphism classes of nilpotent quotients. The central conjecture concerning parafree groups is Baumslag's Parafree Conjecture, which predicts that if G is a finitely generated parafree group, then $\mathrm{H}_2(G; \mathbb{Z}) = 0$. The Strong Parafree Conjecture additionally predicts that $\operatorname{cd}_{\mathbb{Z}}(G) \leqslant 2$; both conjectures are open. It is easy to see from the definition that parafree groups are in fact residually (torsion-free nilpotent), and in particular are residually poly-$\mathbb{Z}$. It actually turns out the parafree groups are RFRS as well [Rei15, Theorem 9.2]. Note that the assumption that G be finitely generated is necessary in the statement of the (Strong) Parafree Conjecture.

Corollary 4.3.15. *Let G be a finitely generated parafree group of cohomological dimension at most two. The following are equivalent:*

(1) *G satisfies the Parafree Conjecture;*
(2) *$b_2^{(2)}(G) = 0$;*
(3) *G is virtually free-by-cyclic;*

(4) *G is free-by-(free nilpotent).*

Proof. (1) ⇒ (2): If the Parafree Conjecture holds, then $b_2(G) = 0$, and therefore $b_2^{(2)}(G) = 0$ by Lemma 4.3.13.

(2) ⇒ (3): This follows from Theorem 4.1.1, Remark 4.1.2, and the fact that parafree groups are RFRS.

(2) ⇒ (4): Since G is parafree, there is a (finitely generated) free group F such that $G/\gamma_n(G) \cong F/\gamma_n(G)$ for all $n \geqslant 0$, where $\gamma_n(G)$ denotes the nth term of the lower central series of G. Note that all of the quotients $F/\gamma_n(F)$ are finitely generated torsion-free nilpotent, and in particular are poly-$\mathbb{Z}$. Thus, Theorem 4.3.9 implies that $\gamma_n(G)$ is free for sufficiently large n. But $G/\gamma_n(G) \cong F.\gamma_n(F)$ is (by definition) the free nilpotent group of rank $\mathrm{rk}(F)$ and of step n. Hence, G is free-by-(free nilpotent).

(3) ⇒ (1): If G is virtually free-by-cyclic, it is coherent by [FH99] and in particular is of finite type. Moreover, because the Abelianisation of G is torsion-free, $b_1(G; k)$ is independent of the coefficient field k. Thus, we have $b_1^{(2)}(G) = b_1(G; k) - 1$ by [BR15, Corollary 8.1] and $b_2^{(2)}(G; k) = 0$ for all fields k. Hence, the Euler characteristic of G is

$$\chi(G) = -b_1^{(2)}(G) = 1 - b_1(G; k) + b_2(G; k)$$

for every field k, which implies $b_2(G; k) = 0$ for all k. Because G is of finite type, this implies that $\mathrm{H}_2(G; \mathbb{Z}) = 0$, as desired.

(4) ⇒ (1): This follows as in the previous paragraph, since we only needed the fact that G was finitely presented, and this is true for finitely generated free-by-(free nilpotent) groups of cohomological dimension at most two by Corollary 4.3.11. □

Remark 4.3.16. It is not known whether finitely generated parafree groups are finitely presented, let alone whether they are coherent. An immediate consequence of Corollary 4.3.15 and Corollary 4.3.11 is that finitely generated parafree groups satisfying the Strong Parafree Conjecture are coherent.

Chapter 5

The Kaplansky Zero Divisor Conjecture for 3-manifold groups

The purpose of this chapter is to show that torsion-free fundamental groups of 3-manifolds satisfy Kaplansky's Zero Divisor Conjecture. More precisely, this means if G is torsion-free and is the fundamental group of some 3-manifold, the group algebra $k[G]$ is a domain for any field k. This will be achieved by proving that $k[G]$ is a subring of a division ring whenever G is torsion-free. The material from this chapter is taken from the article [FSP23] of Sánchez-Peralta and the author.

5.1 Graphs of rings

In this section we introduce *graphs of rings* and prove some of their basic properties. The amalgamated product of rings over a common subring has been studied extensively (see, for instance, [Coh06]) and the HNN extension of rings was defined and studied by Dicks in [Dic83]. The upshot of this section is Corollary 5.1.11, which states that crossed products of graphs of Hughes-free embeddable groups embed in a division ring.

We define graphs of rings in complete analogy with graphs of groups. We take graphs to be connected and oriented, with $\overline{e}$ denoting the same edge as e but with the opposite orientation. Every edge e has an origin vertex $o(e)$ and a terminus vertex $t(e)$ such that $o(e) = t(\overline{e})$. Graphs are allowed to have loops and multiple edges.

Definition 5.1.1 (The graph of rings with respect to a spanning tree)**.** Let Γ be a graph and let T be a spanning tree. For each vertex v of Γ we have a *vertex ring* R_v and for each edge e of Γ we have an *edge ring* R_e and we impose $R_e = R_{\overline{e}}$ for every edge e. Moreover, for each (directed) edge e there is an injective ring homomorphism

$\varphi_e \colon R_e \to R_{t(e)}$. Then the *graph of rings* $\mathscr{R}_{\Gamma,T} = (R_v, R_e)$ is the ring defined as follows:

(1) for each edge of e of Γ we introduce a formal symbols t_e;
(2) $\mathscr{R}_{\Gamma,T}$ is generated by the vertex rings R_v and the elements t_e, t_e^{-1} and subjected to the relations

- $t_{\overline{e}} t_e = t_e t_{\overline{e}} = 1$;
- $t_e \varphi_{\overline{e}}(r) t_{\overline{e}} = \varphi_e(r)$ for all $r \in R_e$;
- if $e \in T$, then $t_e = 1$.

Define $\mathscr{R}^*_\Gamma$ in the same way as $\mathscr{R}_{\Gamma,T}$, except drop the relations $t_e = 1$ if $e \in T$. There is a canonical quotient map $\pi_T \colon \mathscr{R}^*_\Gamma \to \mathscr{R}_{\Gamma,T}$.

Definition 5.1.2 (The based graph of rings)**.** We retain all the notations of Definition 5.1.1. Fix a base vertex $v_0 \in \Gamma$. We say that an element of $\mathscr{R}^*_\Gamma$ is a *loop element* if it is of the form $r_0 t_{e_1} r_1 t_{e_2} \cdots t_{e_n} r_n$ and

(1) $r_0 \in R_{o(e_1)}$;
(2) $r_i \in R_{t(e_i)}$ for all $1 \leqslant i \leqslant n$;
(3) $t(e_i) = o(e_{i+1})$ for all $1 \leqslant i \leqslant n-1$;
(4) $o(e_1) = t(e_n) = v_0$.

We then define $\mathscr{R}_{\Gamma,v_0}$ to be the subring of $\mathscr{R}^*_\Gamma$ generated by the loop elements. Since the product of loop elements is clearly a loop element, $\mathscr{R}_{\Gamma,v_0}$ consists of the elements of $\mathscr{R}^*_\Gamma$ that can be expressed as sums of loop elements.

Remark 5.1.3. When defining a ring with generators and relations, we are quotienting a freely generated ring by an ideal. Thus, with these definitions, we of course run the risk that $\mathscr{R}^*_\Gamma, \mathscr{R}_{\Gamma,T}$, or $\mathscr{R}_{\Gamma,v_0}$ is zero, and that we have lost all information about the vertex and edge rings. This never happens in the graph of groups construction, but not much can be said for a general graph of rings. In the situations of interest, however, we will see that this does not happen, and that the vertex rings inject into the graph of rings (see Lemma 5.1.5 and Proposition 5.1.9) as one would hope.

The following result is the analogue of [Ser77, Ch. 1, §5.2, Proposition 20]. In particular, it implies that the isomorphism types of $\mathscr{R}_{\Gamma,T}$ and $\mathscr{R}_{\Gamma,v_0}$ are independent of the choices of T and v_0, respectively. We will thus simplify the notation and denote the graph of rings by $\mathscr{R}_\Gamma$.

Proposition 5.1.4. *Restricting the canonical projection $\pi_T \colon \mathscr{R}^*_\Gamma \to \mathscr{R}_{\Gamma,T}$ induces an isomorphism $\alpha := \pi_T|_{\mathscr{R}_{\Gamma,v_0}} \colon \mathscr{R}_{\Gamma,v_0} \to \mathscr{R}_{\Gamma,T}$.*

Proof. The proof is analogous to that of [Ser77, Ch. 1, §5.2, Proposition 20], to which we refer the reader for more details. For every vertex v of Γ, let $c_v = e_1 \cdots e_n$ be the geodesic path from v_0 to v in T and let $\gamma_v = t_{e_1} \cdots t_{e_n}$ be the corresponding element of $\mathscr{R}_\Gamma^*$. Put $x' = \gamma_v x \gamma_v^{-1}$ whenever $x \in R_v$ and $t'_e = \gamma_{o(e)} t_e \gamma_{t(e)}^{-1}$ for every edge e of Γ. It is straightforward to show that the assignment $\beta(x) = x'$ and $\beta(t_e) = t'_e$ induces a well-defined homomorphism $\beta\colon \mathscr{R}_{\Gamma,T} \to \mathscr{R}_{\Gamma,v_0}$ such that $\alpha \circ \beta = \mathrm{id}$ and $\beta \circ \alpha = \mathrm{id}$. □

When G decomposes as graph of groups $\mathscr{G}_\Gamma$, the crossed product $k*G$ decomposes as a graph of rings in the expected way.

Lemma 5.1.5. *Let $\mathscr{G}_\Gamma = (G_v, G_e)$ be a graph of groups with fundamental group G and let R be a ring. Then any crossed product $R * G$ decomposes as a graph of rings $\mathscr{R}_\Gamma = (R * G_v, R * G_e)$, where the edge maps $R * G_e \to R * G_{t(e)}$ are induced by the edge maps $G_e \to G_{t(e)}$ of the graph of groups.*

Proof. Let v_0 be a vertex in Γ; we work with the based graph of rings presentation for $\mathscr{R}_\Gamma$. Define a homomorphism $\alpha\colon R * G \to \mathscr{R}_\Gamma$ as follows. Write $g \in G$ as a loop element $g_1 e_1 g_2 e_2 \cdots e_n g_n$ and put $\alpha(g) = g_1 e_1 g_2 e_2 \cdots e_n g_n \in \mathscr{R}_\Gamma$. This defines a homomorphism of G into the unit group $\mathscr{R}_\Gamma^\times$, so α extends to a homomorphism $R * G \to \mathscr{R}_\Gamma$ by R-linearity.

On the other hand if we put $\beta(g_1 e_1 g_2 e_2 \cdots e_n g_n) = g_1 e_1 g_2 e_2 \cdots e_n g_n \in R * G$ for a loop element $g_1 e_1 g_2 e_2 \cdots e_n g_n$, we also obtain a well-defined homomorphism $\beta\colon \mathscr{R}_\Gamma \to R*G$, since the relations in $\mathscr{R}_\Gamma$ hold in $R*G$ (by the based graph of groups presentation for G). □

Definition 5.1.6. Let $\mathscr{G}_\Gamma = (G_v, G_e)$ be a graph of torsion-free groups with fundamental group G and fix a division ring k and a crossed product $k * G$. Then $\mathscr{G}_\Gamma$ is called *$\mathcal{D}$-compatible* if the following conditions are met:

(1) For every vertex v of Γ, there is an embedding $k * G_v \hookrightarrow \mathcal{D}_v$, where $\mathcal{D}_v$ denotes a division ring.

(2) Let $\mathcal{D}_e$ denote $\mathrm{Div}(\varphi_e(k * G_e), \mathcal{D}_{t(e)})$. For all vertices v and all edges e such that $t(e) = v$, any set of right coset representatives of $\varphi_{t(e)}(G_e)$ in G_v is left-linearly independent over $\mathcal{D}_e$.

(3) $\mathcal{D}_e \cong \mathcal{D}_{\bar{e}}$ as $k * G_e$-division rings for every edge e of Γ.

Remark 5.1.7. Condition (2) is automatically satisfied if the embeddings $k*G_v \hookrightarrow \mathcal{D}_v$ are Linnell. If, in addition, the vertex groups are locally indicable, then condition (3) is automatically satisfied by the uniqueness of Hughes-free division rings [Hug70].

In what follows, we will usually (by an abuse of notation) denote the fundamental group of a graph of groups $\mathscr{G}_\Gamma = (G_v, G_e)$ by $\mathscr{G}_\Gamma$; a choice of base vertex in Γ will always be implicit. If $\mathscr{G}_\Gamma = (G_v, G_e)$ is a $\mathcal{D}$-compatible graph of groups, then we can form the *graph of division rings* on Γ with vertex division rings $\mathcal{D}_v$ and edge division rings $\mathcal{D}_e$; we denote it by $\mathscr{D}\mathscr{G}_\Gamma$. Our next goal is to prove that $k * \mathscr{G}_\Gamma$ embeds into $\mathscr{D}\mathscr{G}_\Gamma$. For this, we will need the following normal form theorem.

Theorem 5.1.8 ([Dic83, Theorems 34(i) and 35(i)]).

(1) *Let B and C be rings containing a common subring A such that B (resp. C) is free as a left A-module with basis $\{1\} \sqcup X$ (resp. $\{1\} \sqcup Y$). Then the amalgam $B *_A C$ is free as a left B-module on the set of sequences of strings $y_1x_1y_2x_2\cdots$ with $x_i \in X$ and $y_i \in Y$ not beginning with an element of X and including the empty sequence.*

(2) *Let $B*_A$ be an HNN extension of rings with stable letter t such that B is free as a left A-module under both edge maps, with bases $\{1\} \sqcup X$ and $\{1\} \sqcup Y$. Then $B*_A$ is free as a left B-module on the set of linked expressions constructed from*

$$\begin{array}{cc} \ominus\ X\ \oplus & \ominus\ Xt^{-1} \cup \{t^{-1}\}\ \ominus \\ \oplus\ tY \cup \{t\}\ \oplus & \oplus\ tYt^{-1}\ \ominus \end{array}$$

not beginning with an element of X or Xt^{-1} and including the empty sequence.

A *linked expression* is a word $a_1a_2a_3\cdots$ such that if a_i belongs to a set with a $\oplus$ (resp. $\ominus$) to its right, then a_{i+1} must belong to a set with a $\oplus$ (resp. $\ominus$) to its left; we refer to [Dic83] for a precise definition. Note that (1) is deduced from earlier work of Cohn [Coh59] or [Ber74].

In the proof of the following lemma, all transversals that appear are assumed to contain the relevant group's identity element.

Proposition 5.1.9. *Let k be a division ring and let $\mathscr{G}_\Gamma = (G_v, G_e)$ be a $\mathcal{D}$-compatible graph of groups (for some fixed crossed product $k * \mathscr{G}_\Gamma$). Then the natural map*

$$k * \mathscr{G}_\Gamma \to \mathscr{D}\mathscr{G}_\Gamma$$

is an embedding.

Proof. Write $G = \mathscr{G}_\Gamma$. First assume that Γ is finite. We simultaneously prove the following pair of statements by induction on the number of edges in Γ:

(1) $k * \mathscr{G}_\Gamma \to \mathscr{D}\mathscr{G}_\Gamma$ is an embedding, and

(2) for any vertex v of Γ, there is a right transversal T of G_v in G such that the image of T in $\mathscr{D}\mathscr{G}_\Gamma$ is linearly independent over $\mathcal{D}_v$.

If Γ has no edges, then it consists of a single vertex and the claims are trivial. Now suppose that Γ has at least one edge. Let v be a vertex of Γ and let e be an edge such that $o(e) = v$. Assume that $\Gamma \smallsetminus e$ is disconnected with connected components Γ_1 and Γ_2, where $v \in \Gamma_1$. By induction, $k * \mathscr{G}_{\Gamma_1}$ embeds in $\mathscr{D}\mathscr{G}_{\Gamma_1}$ and there is a right transversal T_1 of G_v in $\mathscr{G}_{\Gamma_1}$ which remains linearly independent over $\mathcal{D}_v$. Let S_1 be a right transversal for (the image of) G_e in G_v. Then S_1 is also linearly independent over $\mathcal{D}_e$ in $\mathcal{D}_v$ by $\mathcal{D}$-compatibility. Thus, S_1T_1 is a right transversal for G_e in $\mathscr{G}_{\Gamma_1}$ and it is linearly independent over $\mathcal{D}_e$ in $\mathscr{D}\mathscr{G}_{\Gamma_1}$.

Moreover, we also have that $k * \mathscr{G}_{\Gamma_2}$ embeds in $\mathscr{D}\mathscr{G}_{\Gamma_2}$. By a similar argument, there is a transversal T_2 for $G_{t(e)}$ in $\mathscr{G}_{\Gamma_2}$ and a transversal S_2 for G_e in $G_{t(e)}$ such that T_2 and S_2T_2 are linearly independent over $\mathcal{D}_{t(e)}$ and $\mathcal{D}_e$, respectively.

Let X be the set of alternating expressions of the form $y_1x_1y_2x_2\cdots$ with $x_i \in S_1T_1$ and $y_i \in S_2T_2$ not beginning with an element of S_1T_1. Note that X is a right transversal for $\mathscr{G}_{\Gamma_1}$ in $\mathscr{G}$. By Theorem 5.1.8(1), we have that $k * G$ embeds in $\mathscr{D}\mathscr{G}_\Gamma$ and X is linearly independent over $\mathscr{D}\mathscr{G}_{\Gamma_1}$. To complete the induction, note that T_1X is a right transversal for G_v in $\mathscr{G}_\Gamma$; so by linear independence of T_1 and X over $\mathcal{D}_v$ and $\mathscr{D}\mathscr{G}_{\Gamma_1}$, respectively, we conclude that T_1X is linearly independent over $\mathcal{D}_v$.

The case where $\Gamma \smallsetminus e$ is connected is proved similarly using Theorem 5.1.8(2); we omit the proof.

We now drop the assumption that Γ is finite. For a contradiction, assume that $k*\mathscr{G}_\Gamma \to \mathscr{D}\mathscr{G}_\Gamma$ is not injective. Let x be a non-trivial element of the kernel and let $\Gamma' \subseteq \Gamma$ on which x is supported. The image of x in $\mathscr{D}\mathscr{G}_\Gamma$ will be a finite linear combination of relators, which are supported in some finite subgraph $\Gamma'' \subseteq \Gamma$. Enlarging Γ' and Γ'' if necessary, we may assume that $\Gamma' = \Gamma''$. But then x is a non-trivial element of the kernel of $k * \mathscr{G}_{\Gamma'} \to \mathscr{D}\mathscr{G}_{\Gamma'}$, a contradiction. $\square$

Recall that a left *free ideal ring* is a ring all of whose left ideals are free of unique rank. A left *semifir* is a ring all of whose finitely generated left ideals are free of unique rank. The semifir property is left-right symmetric, so we drop the left-right specification when discussing these rings and simply refer to them as semifirs. The main results of this section now follows easily from a powerful result of Cohn stating that semifirs embed into (universal) division rings, and from results of Cohn and Dicks, which together imply that a graph of semifirs with division ring edge rings is again a semifir. These results are cited in the proof of the following theorem.

Theorem 5.1.10. *Let k be a division ring and let $\mathscr{G}_\Gamma = (G_v, G_e)$ be a $\mathcal{D}$-compatible graph of groups (for some fixed crossed product $k * \mathscr{G}_\Gamma$). Then $k * \mathscr{G}_\Gamma$ embeds into a division ring.*

Proof. We will prove that $\mathscr{D}\mathscr{G}_\Gamma$ is a semifir, namely that all of its finitely generated left (or right) ideals are free of unique rank. The result then follows from Proposition 5.1.9 and Cohn's theorem stating that every semifir embeds into a division ring [Coh06, Corollary 7.5.14].

We begin with the case that Γ is finite. This follows by induction on the number of edges, using the facts that amalgams of semifirs over a division ring and HNN extensions of a semifir over a division ring are still semifirs ([Coh59] and [Dic83, Theorems 34(ii) and 35(ii)].

If Γ is infinite, then the result follows since $\mathscr{D}\mathscr{G}_\Gamma$ is the colimit of the semifirs $\mathscr{D}\mathscr{G}_{\Gamma'}$ with Γ' finite (see [Coh85, §1.1 Exercise 3]). □

Corollary 5.1.11. *Let k be a division ring, let $\mathscr{G}_\Gamma = (G_v, G_e)$ be a graph of locally indicable groups, and fix a crossed product $k * \mathscr{G}_\Gamma$. Suppose there is a Hughes-free embedding $k * G_v \hookrightarrow \mathcal{D}_{k*G_v}$ for each vertex v of Γ. Then $k * \mathscr{G}_\Gamma$ embeds in a division ring.*

Proof. By Theorem 2.3.10, Hughes-free embeddings are in fact Linnell embeddings [Grä20, Corollary 8.3]. Recall that if $A \leqslant B$ are groups and there is a Hughes-free embedding $k * B \hookrightarrow \mathcal{D}_{k*B}$, then $\mathrm{Div}(k * A, \mathcal{D}_{k*B})$ is isomorphic to the unique Hughes-free division ring $\mathcal{D}_{k*A}$ (this follows from the uniqueness of Hughes-free division rings [Hug70]). Thus, $\mathscr{G}_\Gamma$ is $\mathcal{D}$-compatible. □

Corollary 5.1.12. *Let k be a subfield of $\mathbb{C}$ and let $\mathscr{G}_\Gamma = (G_v, G_e)$ be a graph of (torsion-free groups satisfying the Strong Atiyah Conjecture over k). Then the group algebra $k\mathscr{G}_\Gamma$ embeds in a division ring.*

Proof. Since the vertex groups satisfy the Strong Atiyah Conjecture, kG_v embeds into the Linnell division ring $\mathcal{D}(G_v)$. The fact that $\mathscr{G}_\Gamma$ is $\mathcal{D}$-compatible follows from the fact that if B is a torsion-free group satisfying the Strong Atiyah Conjecture and $A \leqslant B$, then the division closure of kA in $\mathcal{D}(B)$ is isomorphic to $\mathcal{D}(A)$ (see either [Kie20a, Proposition 4.6] or [Lüc02, Chapter 10]). □

We conclude the section with a short application of our main result. Recall that Higman's group H can be defined by the presentation

$$\langle a, b, c, d \mid b^a = b^2, c^b = c^2, d^c = d^2, a^d = a^2 \rangle.$$

It can also be realised as a square of groups with $\mathrm{BS}(1,2)$ vertex groups, $\mathbb{Z}$ edge groups, and trivial face group. The group H was constructed by Higman in [Hig51], and it was the first example of an infinite group with no non-trivial finite quotients. It also has an infinite simple quotient. While Higman's group often serves as a source of counterexamples, we will see here that it's group algebras are quite well behaved, at least for fields of characteristic zero.

Rivas and Triestino showed that Higman's group acts faithfully and continuously on $\mathbb{R}$, and therefore is left-orderable and in particular $R[H]$ satisfies Kaplansky's Zero Divisor Conjecture for all domains R [RT19, Theorem A, Corollary B]. Here we show that $k[H]$ has the (a priori) stronger property of embedding into a division ring, at least when k is a field of characteristic zero.

Proposition 5.1.13. *Let k be a field of characteristic zero. Then $k[H]$ embeds into a division ring.*

Proof. Indeed, note that H decomposes as $H = G_1 *_A G_2$, where

$$G_1 = \langle a, b, c \rangle, \quad A = \langle a, c \rangle \cong F_2, \quad G_2 = \langle a, c, d \rangle.$$

Moreover, for $i = 1, 2$, we have that $G_i \cong \mathrm{BS}(1,2) *_{\mathbb{Z}} \mathrm{BS}(1,2)$ is a cyclic amalgam of locally indicable groups, and therefore is locally indicable by a result of Howie [How82, Theorem 4.2]. Hence, $k[G_i]$ has a Hughes-free embedding for $i = 1, 2$ by [JZLÁ20, Corollary 1.4] and therefore $k[H]$ embeds into a division ring by Corollary 5.1.11. □

We obtain a consequence on $k[H]$ that does not follow from the fact that it has no zero-divisors, and requires the embedding into a division ring. A ring R is *stably finite* if for any pair of square matrices A and B, we have $AB = 1$ if and only if $BA = 1$. Being stably finite is clearly a property that passes to subrings, and it also clearly holds for division rings.

Corollary 5.1.14. *Let H be Higman's group defined above. The group algebra $k[H]$ is stably finite whenever k is a field of characteristic zero.*

5.2 Virtually compact special groups

Fix an arbitrary division ring k for the remainder of the section. Our goal is to prove that if G is a torsion-free virtually compact special group, then any crossed product $k * G$ admits a unique Linnell embedding. More generally, we will obtain the same conclusion under the assumptions that G is a torsion-free extension of a compact special group by an elementary amenable group. Our arguments are based on those of Linnell and Schick [LS07, Corollary 4.62] combined with those of Schreve [Sch14]. We begin with two preliminary results.

Lemma 5.2.1. *Let $k * G$ be a crossed product of a division ring k and a torsion-free group G. Suppose there is a normal subgroup $H \trianglelefteq G$ such that G/H is elementary amenable and such that there exists a Linnell embedding $k*H \hookrightarrow \mathcal{D}$. If the conjugation action of G on H extends to an action on $\mathcal{D}$ and $\mathcal{D} * [G/H]$ is a domain, then the embedding $k * G \hookrightarrow \mathrm{Ore}(\mathcal{D} * [G/H])$ is Linnell.*

Proof. First note that $\mathcal{D} * [G/H]$ is an Ore domain by [LS07, Lemma 2.5]. For the sake of brevity, if $A \leqslant G$, then we write $\mathcal{D}_A$ for the division closure of $k * A$ in $\mathrm{Ore}(\mathcal{D} * [G/H])$ and we note that $\mathcal{D}_A = \mathrm{Ore}(\mathcal{D}_{H \cap A} * [A/H \cap A])$. Let $N \leqslant G$ be a subgroup, let $t_1, \dots, t_n$ be distinct right N-coset representatives in G, and let $\alpha_1, \dots, \alpha_n \in \mathcal{D}_N$ be such that

$$\alpha_1 t_1 + \cdots + \alpha_n t_n = 0.$$

By multiplying on the left by a common denominator, we may assume that $\alpha_i \in \mathcal{D}_{H \cap N} * [N/H \cap N]$. Fixing a collection $s_1, \dots, s_k$ of right $H \cap N$-coset representatives in N, for each i we can write $\alpha_i = \sum_{l=1}^k \beta_l^i s_l$ for some $\beta_l^i \in \mathcal{D}_{H \cap N}$. The previous line becomes

$$\sum_{l=1}^{k} (\beta_l^1 s_l t_1 + \cdots + \beta_l^n s_l t_n) = 0.$$

Observe that the elements $s_l t_m$ lie in different $H \cap N$-cosets, so the previous line has the form

$$\gamma_1 r_1 + \cdots + \gamma_j r_j = 0$$

where $\gamma_d \in \mathcal{D}_{H \cap N}$ for each d and $r_1, \dots, r_j$ is a collection of distinct $H \cap N$-coset representatives (here, $j = kn$ and each element γ_d is equal to some β_l^i). Since the H-cosets are left-linearly independent over $\mathcal{D} = \mathcal{D}_H$ by assumption, it suffices to consider the case where the elements r_d are all contained in the same H-coset. But then there is some $g \in G$ such that $r_d = h_d g$ for all d, where the elements $h_d \in H$ lie in different $H \cap N$-cosets. We obtain $\sum_d \gamma_d h_d = 0$, implying that $\gamma_d = 0$ for all d by the Linnell property. But then $\alpha_i = 0$ for each $1 \leqslant i \leqslant n$, as desired. □

The main situation where we will use the previous lemma is when H is locally indicable and Hughes-free embeddable.

Corollary 5.2.2. *Let $k * G$ be a crossed product of a torsion-free group G and a division ring k. Suppose that $H \trianglelefteq G$ is a normal and locally indicable subgroup such that G/H is elementary amenable. If there is a Hughes-free embedding $k * H \hookrightarrow \mathcal{D}_{k*H}$ and $\mathcal{D}_{k*H} * [G/H]$ is a domain, then embedding $k * G \hookrightarrow \operatorname{Ore}(\mathcal{D}_{k*H} * [G/H])$ is Linnell and is unique among Linnell embeddings up to $k * G$-isomorphism.*

Proof. If $\mathcal{D}_{k*H}$ exists, then it is Linnell by Theorem 2.3.10. By Lemma 5.2.1, the embedding $k * G \hookrightarrow \operatorname{Ore}(\mathcal{D}_{k*H} * [G/H])$ is Linnell. Now suppose that $k * G \hookrightarrow \mathcal{D}$ is Linnell. Then $\operatorname{Div}(k * H, \mathcal{D})$ is Hughes-free, and therefore isomorphic to $\mathcal{D}_{k*H}$ by [Hug70]. Since $\mathcal{D}$ is Linnell, the cosets of G/H are left-linearly independent over $\mathcal{D}_{k*H}$ and therefore there is an embedding $\mathcal{D}_{k*H} * [G/H] \hookrightarrow \mathcal{D}$, where G acts by conjugation on $\mathcal{D}_{k*H}$. By the universal property of Ore localisations, there is a homomorphism $\operatorname{Ore}(\mathcal{D}_{k*H} * [G/H]) \hookrightarrow \mathcal{D}$, which is surjective since $k * G$ generates $\mathcal{D}$ as a division ring. This proves the uniqueness statement. □

A key tool in our arguments is the factorisation property introduced by Schreve.

Definition 5.2.3 ([Sch14])**.** A group G has the *factorisation property* if for every finite group Q and every epimorphism $G \to Q$, there is a torsion-free elementary amenable group E such that the previous map factors as $G \to E \to Q$.

We briefly recall the notion of goodness (in the sense of Serre), which relates the cohomology of a group G with that of its profinite completion. More specifically, G is *good* if $\mathrm{H}^n(G; M) \cong \mathrm{H}^n_{\mathrm{cts}}(\widehat{G}; M)$ for all finite G-modules M and all $n \geqslant 0$ (see [Ser97, I.2.6, Exercise 2]). Here, $\widehat{G}$ denotes the profinite completion of G, that is, the inverse limit of the directed system of all finite quotients of G. The cohomology $\mathrm{H}^n_{\mathrm{cts}}$ is the *continuous cohomology* of $\widehat{G}$, which is the cohomology of $\widehat{G}$ with respect to the continuous cochains. We will only use goodness when applying the following result of Friedl, Schreve, and Tillmann: if G is a finitely generated good group of finite cohomological dimension, and G virtually has the factorisation property, then G has the factorisation property [FST17, Theorem 3.7]. Many groups are known to be good; importantly for us, virtually compact special groups are good [Sch14, Corollary 4.3], as are all fundamental groups of compact 3-manifolds [Cav12, Section 3.5].

Theorem 5.2.4. *Let H be a locally indicable good group of finite type with the factorisation property and let $1 \to H \to G \to Q \to 1$ be a group extension with G torsion-free and Q finite. If k is a division ring such that there is a Hughes-free embedding of $k * H$ into $\mathcal{D}_{k*H}$, then $k * G$ has a unique Linnell embedding into a division ring.*

Proof. By [FST17, Theorem 3.7], G has the factorisation property and therefore there is a normal subgroup $U \trianglelefteq G$ such that $U \leqslant H$ and G/U is torsion-free and elementary amenable. By the uniqueness of Hughes-free embeddings, we can form each of the following rings:

$$\mathcal{D}_{k*U} * [H/U], \quad \mathcal{D}_{k*U} * [G/U], \quad \mathcal{D}_{k*H} * [G/H].$$

Since H/U and G/U are torsion-free elementary amenable, $\mathcal{D}_{k*U} * [H/U]$ and $\mathcal{D}_{k*U} * [G/U]$ are Ore domains by [LS07, Lemma 2.5] and so the diagram

$$\begin{array}{ccc} \mathcal{D}_{k*U} * [H/U] & \hookrightarrow & \mathcal{D}_{k*U} * [G/U] \\ \downarrow & & \downarrow \\ \mathrm{Ore}(\mathcal{D}_{k*U} * [H/U]) & \hookrightarrow & \mathrm{Ore}(\mathcal{D}_{k*U} * [G/U]) \end{array}$$

commutes. By Hughes-freeness of $\mathcal{D}_{k*H}$, the map $\mathcal{D}_{k*U} * [H/U] \to \mathcal{D}_{k*H}$ is an injection. This implies that $\mathrm{Ore}(\mathcal{D}_{k*U} * [H/U]) \cong \mathcal{D}_{k*H}$ by the universal property of Ore localisation.

Consider the following diagram:

$$\begin{array}{ccccc} \mathcal{D}_{k*H} & \overset{\cong}{\longrightarrow} & \mathrm{Ore}(\mathcal{D}_{k*U} * [H/U]) & \hookrightarrow & \mathrm{Ore}(\mathcal{D}_{k*U} * [G/U]) \\ \downarrow & & \downarrow & & \Big| \cong \\ \mathcal{D}_{k*H} * [G/H] & \overset{\cong}{—} & \mathrm{Ore}(\mathcal{D}_{k*U} * [H/U]) * [G/H] & \overset{\cong}{—} & \mathrm{Ore}((\mathcal{D}_{k*U} * [H/U]) * [G/H]) \,. \end{array}$$

The left and middle vertical maps are the obvious inclusions and the right vertical map is a standard isomorphism of crossed products. The two left isomorphisms come from the isomorphism $\mathrm{Ore}(\mathcal{D}_{k*U} * [H/U]) \cong \mathcal{D}_{k*H}$ discussed above. For the bottom right isomorphism, it is not hard to show that the natural map

$$\mathrm{Ore}(\mathcal{D}_{k*U} * [H/U]) * [G/H] \to \mathrm{Ore}((\mathcal{D}_{k*U} * [H/U]) * [G/H]),$$

is injective. Therefore $\mathrm{Ore}(\mathcal{D}_{k*U} * [H/U]) * [G/H]$ is a domain, which implies it is a division ring since G/H is finite. This proves that $\mathcal{D}_{k*H} * [G/H]$ is a division ring, which clearly contains $k * G$. Moreover, the embedding is Linnell and unique among Linnell embeddings by Corollary 5.2.2. □

Corollary 5.2.5. *Let H be a locally indicable good group of finite type with the factorisation property and let $1 \to H \to G \to A \to 1$ be a group extension with G torsion-free and A elementary amenable. If k is a division ring such that there is a Hughes-free embedding of $k * H$ into $\mathcal{D}_{k*H}$, then $k * G$ embeds into a division ring.*

Proof. By Hughes-freeness, the twisted action of A on $k * H$ extends to a twisted action of A on $\mathcal{D}_{k*H}$. Moreover, $\mathcal{D}_{k*H} * Q$ is a domain for every finite subgroup Q of A by the proof of Theorem 5.2.4. Thus, according to [LS07, Lemma 2.5] $\mathcal{D}_{k*H} * A$ is an Ore domain. Therefore, $\mathcal{D}_{k*H} * A$, and hence also $(k * H) * A \cong K * G$, embeds into a division ring. □

We can now prove the main result of this section. It applies, in particular, to virtually compact special groups G.

Corollary 5.2.6. *Let G is a torsion-free group with a normal virtually compact special group H such that G/H is elementary amenable and k a division ring. Any crossed product $k * G$ has a unique Linnell embedding into a division ring $\mathcal{D}$. Moreover, if H is a normal, finite index, compact special subgroup of G, then the diagram*

$$\begin{array}{ccccc} k * H & \hookrightarrow & (k * H) * [G/H] & \xrightarrow{\cong} & k * G \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{D}_{k*H} & \hookrightarrow & \mathcal{D}_{k*H} * [G/H] & \xrightarrow{\cong} & \mathcal{D} \end{array}$$

commutes.

Proof. Compact special groups are residually (torsion-free nilpotent) and therefore $\mathcal{D}_{k*H}$ exists by [JZ21, Theorem 1.1]. Moreover, compact special groups are good and have the factorisation property [Sch14, Corollary 4.3] and are of finite type, since they have finite classifying spaces. Hence there is an embedding of $k * G$ into a division ring $\mathcal{D} \cong \mathcal{D}_{k*H} * [G/H]$ by Theorem 5.2.4. The embedding $k * G \hookrightarrow \mathcal{D}$ is Linnell and unique by Corollary 5.2.2. □

5.3 3-manifold groups

Fix an arbitrary division ring k. The goal of this section is to prove that twisted group algebras of torsion-free 3-manifold groups embed into division rings. We will make repeated use of the Scott Core Theorem, which is stated below.

Theorem 5.3.1 (Scott Core Theorem [Sco73])**.** *Let M be a 3-manifold with finitely generated fundamental group. There is a compact submanifold N of M such that the inclusion $N \hookrightarrow M$ induces an isomorphism $\pi_1(N) \xrightarrow{\cong} \pi_1(M)$.*

Our proof will also rely on the Prime and JSJ Decomposition Theorems for 3-manifolds and on the graph of rings construction discussed in the previous section.

Since we do not want to assume M is orientable, we will need statements of these decomposition theorems in the non-orientable case. The statements will be as found in Bonahon's survey [Bon02, Theorems 3.1, 3.2, and 3.4], so we take some care to ensure that the definitions recalled in the next paragraph coincide with Bonahon's.

An embedded copy of S^2 in M is *essential* if no component of $M \smallsetminus S^2$ is homeomorphic to a 3-ball. An embedded copy of $\mathbb{R}P^2$ in M is *essential* if $M \smallsetminus \mathbb{R}P^2$ has two components. Then M is *irreducible* if it contains no essential spheres or projective planes. An embedded copy of the torus T^2 in M is *essential* if the induced map $\pi_1(T^2) \to \pi_1(M)$ is injective. An embedded Klein bottle K in M is *essential* if the map $\pi_1(T^2) \to \pi_1(K) \to \pi_1(M)$ is injective, where the first map is induced from the double cover $T^2 \to K$.

Since we always assume that $\pi_1(M)$ is torsion-free, we will actually not need to worry about embedded copies of $\mathbb{R}P^2$, as the following lemma shows.

Lemma 5.3.2. *If M is a 3-manifold and $\mathbb{R}P^2 \hookrightarrow M$ is an embedding, then the induced homomorphism $\pi_1(\mathbb{R}P^2) \to \pi_1(M)$ is injective.*

Proof. Identify $\mathbb{R}P^2$ with its image in M. Fix an exhaustion of M by compact 3-dimensional submanifolds all containing $\mathbb{R}P^2$, say $M = \bigcup_{i \in I} M_i$. It will be sufficient to prove that $\pi_1(\mathbb{R}P^2) \to \pi_1(M_i)$ is non-trivial (and hence injective) for each $i \in I$, since $\pi_1(M) \cong \varinjlim_{i \in I} \pi_1(M_i)$.

Fix some $i \in I$, let M_i' be a copy of M_i, and let $N = M_i \cup_{\partial M_i} M_i'$ denote the double of M_i over its boundary. Observe that $\pi_1(M_i) \leqslant \pi_1(N)$, since M_i is a retract of N. We have $\mathbb{R}P^2 \subseteq M_i \subseteq N$.

The lifts of the embedded $\mathbb{R}P^2$ to the universal cover $\widetilde{N}$ of N form a collection of 2-spheres or projective planes in $\widetilde{N}$. In [Sam69], Samelson proves that smooth hypersurfaces in $\mathbb{R}^n$ must be orientable, and remarks at the end of his note that the proof extends to the case of hypersurfaces in simply connected manifolds. Hence, the embedded $\mathbb{R}P^2$ lifts to a collection of 2-spheres in the universal cover. But then a loop in N representing the generator of $\pi_1(\mathbb{R}P^2)$ has a lift to a non-closed path in $\widetilde{N}$, and therefore the loop is non-trivial in $\pi_1(N)$. In particular, the same loop is non-trivial in $\pi_1(M_i)$, as desired. □

We also have the following alternative definition of an embedded Klein bottle, which will be useful for the group-theoretic non-orientable JSJ Decomposition Theorem given below.

Lemma 5.3.3. *An embedded Klein bottle $K \subseteq M$ is essential if and only if the induced map $\pi_1(K) \to \pi_1(M)$ is injective.*

Proof. The covering map is $T^2 \to K$ is π_1-injective, so if the embedding $K \subseteq M$ is π_1-injective, then so is the map $T^2 \to K \hookrightarrow M$.

Let $\iota\colon K \hookrightarrow M$ be the embedding and assume that $T^2 \to K \hookrightarrow M$ is π_1-injective. Then either $\iota_*\pi_1(K)$ contains $\mathbb{Z}^2$ as an index 2 subgroup in which case ι is π_1-injective, or $\iota_*\pi_1(K) \cong \mathbb{Z}^2$. But $\mathrm{H}_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, which rules out the second case. □

The (non-orientable) Prime Decomposition Theorem states that every closed 3-manifold M can be cut along a collection of pairwise disjointly embedded essential spheres and projective planes such that the complementary components are all irreducible or homeomorphic to $S^2 \times S^1$ minus an open 3-ball (unless $M \cong S^2 \times S^1$ to begin with). We refer the reader to [Bon02, Theorems 3.1 and 3.2] for precise statements. Since we are assuming $\pi_1(M)$ to be torsion-free, there are no copies of $\mathbb{R}P^2$ to cut along by Lemma 5.3.2. This leads to the following group-theoretic version of the Prime Decomposition Theorem for 3-manifolds with torsion-free fundamental group.

Theorem 5.3.4 (Group-theoretic Prime Decomposition Theorem)**.** *Let M be a closed 3-manifold such that $\pi_1(M)$ is torsion-free. Then there is a free group F and finitely many compact, irreducible 3-manifolds $M_1, \ldots, M_n$ each with (possibly empty) incompressible boundary such that*

$$G \cong F * \pi_1(M_1) * \cdots * \pi_1(M_n).$$

Note that we have used Theorem 5.3.1 to formulate the statement in this way.

The non-orientable JSJ Decomposition Theorem states that any irreducible 3-manifold M can be cut along a finite collection of essential pairwise disjointly embedded 2-tori and Klein bottles in such a way that each complementary component is either Seifert fibred or hyperbolic (see [Bon02, Theorem 3.4]). This leads to the following group-theoretic statement of the JSJ Decomposition Theorem.

Theorem 5.3.5 (Group-theoretic JSJ Decomposition Theorem)**.** *Let M be a closed, irreducible 3-manifold. Then $\pi_1(M)$ splits as a graph of groups, where each vertex group is the fundamental group of a hyperbolic or Seifert fibred 3-manifold and each edge group is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}$ (where the extension may be trivial).*

The following lemma is well known; we include a proof because we have not seen it appear without the assumption of orientability. The proof is similar to that of [How82, Theorem 6.1], the only difference being the use of $\mathbb{Z}/2$ coefficients to apply Poincaré duality isomorphisms for non-orientable manifolds.

Lemma 5.3.6. *If M is an aspherical 3-manifold such that $\operatorname{cd}(\pi_1(M)) \leqslant 2$, then $\pi_1(M)$ is locally indicable.*

Proof. Let G be a non-trivial finitely generated subgroup of $\pi_1(M)$ and suppose for a contradiction that $\mathrm{H}_1(G;\mathbb{Z}) = 0$. By the Universal Coefficient Theorem,

$$\mathrm{H}_1(G;\mathbb{Z}/2) = \mathrm{H}^1(G;\mathbb{Z}/2) = 0$$

as well.

Let $\widehat{M} \to M$ be the covering corresponding to G and let $N \subseteq \widehat{M}$ be a compact submanifold such that $\pi_1(N) \cong G$ and the inclusion $N \hookrightarrow \widehat{M}$ induces a π_1-isomorphism. By Poincaré duality, $\mathrm{H}^2(N, \partial N; \mathbb{Z}/2) \cong \mathrm{H}_1(N;\mathbb{Z}/2) = 0$, and the long exact sequence of the pair $(N, \partial N)$ contains the sequence

$$\mathrm{H}^1(N;\mathbb{Z}/2) \to \mathrm{H}^1(\partial N;\mathbb{Z}/2) \to \mathrm{H}^2(N, \partial N;\mathbb{Z}/2),$$

so $\mathrm{H}^1(\partial N;\mathbb{Z}/2) = 0$. Hence, the components of ∂N must all be 2-spheres, which we denote by $S_1, \dots, S_n$.

Since each S_i is nullhomotopic in $\widehat{M}$ (because $\pi_2(\widehat{M}) \cong \pi_2(M) = 0$), each S_i bounds a 3-ball B_i (see, e.g., [Rub97]). We remark that in the proof of [How82, Theorem 6.1], Howie concludes that S_i bounds a "fake 3-cell", which is a compact contractible 3-manifold with 2-sphere boundary; the resolution of the Poincaré Conjecture implies that such manifolds are actually 3-balls.

Note that N is not contained in any of the B_i's, for if this was the case the isomorphism $\pi_1(N) \to \pi_1(\widehat{M})$ would factor through $\pi_1(B_i) = 1$ for some i, contradicting the assumption that G is non-trivial. Hence, $\widehat{M} \cong N \cup D_1 \cup \cdots \cup D_n$ is a closed aspherical 3-manifold. But then $\operatorname{cd}(\pi_1(\widehat{M})) = \operatorname{cd}(G) = 3$, a contradiction. □

The locally indicable case of the main result (Theorem 1.3.3) is more straightforward than the general case and does not rely on the graph of rings construction, so we record it separately. This case will also be used in the proof of the general case below.

Theorem 5.3.7. *If M is a 3-manifold with locally indicable fundamental group, then any twisted group algebra $k * \pi_1(M)$ admits a Hughes-free embedding.*

Proof. To prove that a Hughes-free embedding exists, it is sufficient to prove that $k * H$ admits a Hughes-free embedding for any finitely generated subgroup H of $\pi_1(M)$. Hence, we may assume $\pi_1(M)$ is finitely generated, and therefore that there exists an epimorphism $\pi_1(M) \to \mathbb{Z}$. Let $\widehat{M}$ denote the infinite cyclic cover. We have $\mathrm{cd}(\pi_1(\widehat{M})) \leqslant 2$. By [KL24, Theorem 1.1], $\pi_1(\widehat{M})$ is locally (virtually free-by-cyclic). Free-by-cyclic groups have the factorisation property by [FST17, Lemma 2.4]. Since the finitely generated subgroups of $\pi_1(\widehat{M})$ are fundamental groups of compact 3-manifold groups, $\pi_1(\widehat{M})$ is locally good by [Cav12, Section 3.5]. Hence, $k * H$ admits a Hughes-free embedding for every finitely generated subgroup $H \leqslant \pi_1(\widehat{M})$ by Theorem 5.2.4. Hence, $k * \pi_1(\widehat{M})$ admits a Hughes-free embedding. Finally the twisted group algebra of $\pi_1(M) \cong \pi_1(\widehat{M}) \rtimes \mathbb{Z}$ admits a Hughes-free embedding by [Hug72]. □

We are now ready to prove the general case.

Theorem 5.3.8. *Let M be a 3-manifold, let k be a division ring, and let $k * \pi_1(M)$ be a twisted group algebra. If $\pi_1(M)$ is torsion-free, then $k * \pi_1(M)$ embeds into a division ring.*

Proof. We begin by making some simplifying assumptions. The proof of the following claim is essentially the same as [Kie20a, Proposition 4.11 (3)].

Claim 5.3.9. *It is enough to prove the theorem in the case where M is compact.*

Proof. Since M is a manifold, $\pi_1(M)$ is countable. Thus, we can write $\pi_1(M)$ as an increasing countable union $\bigcup_{n\in\mathbb{N}} H_n$ of finitely generated subgroups $H_n \leqslant \pi_1(M)$. Note that each H_n is itself the fundamental group of a compact 3-manifold by the Scott Core Theorem. For each $n \in \mathbb{N}$, assume there is an embedding $k * H_n \hookrightarrow \mathcal{D}_n$, where $\mathcal{D}_n$ is a division ring. Put a non-principal ultrafilter ω on $\mathbb{N}$ and consider the ultraproduct $\mathcal{D} := \prod_\omega \mathcal{D}_n$, which is also a division ring. There is a map $k * G \to \mathcal{D}$ which sends an element $x \in k * G$ to the class of $(x_n)_{n\in\mathbb{N}}$, where x_n is the image of x in $\mathcal{D}_n$ if x is supported in H_n, and is zero otherwise. It is easily verified that this is an injective ring homomorphism. ◇

From now on we assume that M is compact.

Claim 5.3.10. *It is enough to prove the theorem in the case where M is closed.*

Proof. If M is not closed, let $N = M \cup_{\partial M} M$ be the double of M along its boundary. Then $\pi_1(M) \leqslant \pi_1(N)$, since M is a retract of N. Hence, if $k * \pi_1(N)$ embeds into a division ring, then so does $k * \pi_1(M)$. ◇

Claim 5.3.11. *It is enough to prove the theorem in the case where M is closed and irreducible.*

Proof. By the previous claim, we may assume that M is closed. By Theorem 5.3.4, $\pi_1(M)$ splits as a free product of fundamental groups of irreducible 3-manifolds with (possibly empty) incompressible boundary and a free group. If the twisted group ring of each free factor embeds into a division ring, then so does $k * \pi_1(M)$ by Theorem 5.1.10.

The previous paragraph shows that we may suppose that M is irreducible; we claim it is enough to assume that M is closed and irreducible. Indeed, if M has non-empty incompressible boundary then $\mathrm{cd}(\pi_1(M)) \leqslant 2$, and therefore $k * \pi_1(M)$ embeds into a division ring by Lemma 5.3.6 and Theorem 5.3.7. $\diamond$

Hence, we assume that M is a closed, irreducible 3-manifold. By Theorem 5.3.5, $\pi_1(M)$ splits as a graph of fundamental groups of (hyperbolic of Seifert fibred) 3-manifolds with $\mathbb{Z} \rtimes \mathbb{Z}$ edge groups. Suppose that at least one of the vertex groups is hyperbolic; thus, M is either a mixed 3-manifold or a hyperbolic 3-manifold. In either case, M is virtually special by [Ago13] or [PW18], and therefore virtually fibres over the circle by [Ago08]. Note that surface-by-cyclic groups have the factorisation property by [FST17, Proposition 3.6]. Thus, $\pi_1(M)$ is good and virtually has the factorisation property, so we conclude that $k * \pi_1(M)$ embeds into a division ring by Theorem 5.2.4.

We are left with the case that every vertex group is the fundamental group of a Seifert fibred 3-manifold (i.e. the case where M is a graph manifold). First assume that the graph of groups contains a single vertex group with no edge groups. By [FST17, Lemma 3.9], $\pi_1(M)$ has the factorisation property, and again using the goodness of $\pi_1(M)$ we conclude that $k * \pi_1(M)$ admits an embedding into a division ring.

Now assume that the graph of groups contains multiple vertex groups. In this case, each vertex group is the fundamental group of a Seifert fibred 3-manifold with non-empty toroidal boundary. In particular, each vertex group is locally indicable and therefore admits a Hughes-free embedding. It then follows from Theorem 5.1.10 that $k * \pi_1(M)$ admits an embedding into a division ring. $\square$

Appendix A

Computations

In the first appendix, we present examples of locally indicable groups embedding into division rings and computations of their k-L^2-Betti numbers (k will always denote a field in what follows).

Example A.1 (Amenable groups)**.** If G is amenable and $k[G]$ has no zero divisors, then $k[G]$ is an Ore domain by Kielak's appendix to [Bar19]. Since Ore localisation is a flat functor (just as in the commutative setting), we find that $\mathrm{H}_n(G; \mathrm{Ore}(k[G])) = 0$ for all n, assuming that G is non-trivial.

If $k[G] \hookrightarrow \mathcal{D}$ is an arbitrary embedding into a division ring, then it factors as $k[G] \hookrightarrow \mathrm{Ore}(k[G]) \hookrightarrow \mathcal{D}$ by the universal property of Ore localisation. Since extensions of division rings are flat, we also have $\mathrm{H}_n(G; \mathcal{D}) = 0$ for all n, again assuming G to be non-trivial.

Thus, if G is a non-trivial locally indicable amenable group, then

$$\mathrm{Ore}(k[G]) \cong \mathcal{D}_{k[G]}$$

exists and $b_n^{(2)}(G; k) = 0$ for all fields k and all $n \geqslant 0$.

Example A.2 (Free groups)**.** Let F be the free group on the set S and let $k[F] \hookrightarrow \mathcal{D}$ be any embedding into a division ring. There is a free resolution

$$0 \to \bigoplus_{s \in S} k[F]e_s \to k[F] \to k \to 0$$

where $\bigoplus_{s \in S} k[F]e_s \to k[F]$ is determined by $e_s \mapsto s - 1$, and $k[F] \to k$ is the augmentation map. It follows that the induced map $\bigoplus_{s \in S} \mathcal{D}e_s \to \mathcal{D}$ is non-trivial, and hence is surjective. In particular, the $\mathcal{D}$-dimension of its kernel is $\mathrm{rk}(F) - 1$, so $\mathrm{H}_1(F; \mathcal{D}) = \mathcal{D}^{\oplus(\mathrm{rk}(F)-1)}$ and $\mathrm{H}_1(F; \mathcal{D}) = 0$ otherwise.

Note that free groups are residually torsion-free nilpotent, so the Hughes-free division ring $\mathcal{D}_{k[G]}$ exists for all fields k. Hence, $b_1^{(2)}(F;k) = \mathrm{rk}(F) - 1$ and $b_n^{(2)}(F;k) = 0$ for all $n \neq 1$.

Example A.3 (Surface groups)**.** Let Σ_g be a closed, orientable surface of genus g. Since $\pi_1(\Sigma_g)$ is residually torsion-free nilpotent, the Hughes-free division ring $\mathcal{D}_{k[G]}$ exists for all fields k. The kernel N of any epimorphism $\pi_1(\Sigma_g) \to \mathbb{Z}$ is free, and in particular, has vanishing k-L^2-Betti numbers above dimension 1. Then $b_i^{(2)}(\pi_1(\Sigma_g);k) = 0$ for all $i \geqslant 2$ by Theorem 2.5.5. Hence, $b_1^{(2)}(\pi_1(\Sigma_g);k) = 2g-2$. We will see below how this generalises to other one-relator groups (without assuming that the embedding into a division ring is Hughes-free).

Example A.4 (Fibred groups)**.** Suppose that G admits a virtual map onto $\mathbb{Z}$ with kernel of finite type. Then Theorem 2.5.5 implies that the k-L^2-Betti numbers of G are all zero, provided they are defined. For example, if G is of the form $F_n \rtimes \mathbb{Z}$, then $\mathcal{D}_{k[G]}$ exists for every field k by [Hug72], and thus G is k-L^2-acyclic. Suppose that G is the fundamental group of a finite-volume hyperbolic 3-manifold. Then G is virtually of the form $\pi_1(\Sigma) \rtimes \mathbb{Z}$ for some surface (possibly with boundary) Σ by the work of Agol and Wise [Ago13, Wis21], and $\mathcal{D}_{k[H]}$ exists for any finite-index subgroup $H \leqslant G$ by Theorem 5.3.8.

Example A.5 (One-relator groups)**.** A group is a *one-relator* group if it admits a presentation of the form $G = F(S)/\langle\!\langle r \rangle\!\rangle$, where $F(S)$ is the free group on S and $r \in F(S)$. If r is not a proper power, then G is locally indicable [Bro84], and therefore $\mathcal{D}_{k[G]}$ exists for any field of characteristic zero by Theorem 2.5.3. In general, it is known that $k[G]$ embeds into a division ring for all division rings k by [LL78], but it is not known whether the embedding is Hughes-free. Note that L^2-Betti numbers of arbitrary one-relator groups were originally computed by Dicks and Linnell [DL07].

Let G be a torsion-free one-relator group, let k be a division ring, and let $\mathcal{D}$ be any division ring that contains $k[G]$. Suppose that $r \in F(S)$ is a non-trivial, cyclically reduced word. Then the presentation complex of $\langle S \mid r \rangle$ is aspherical by [Lyn50] and thus yields a free resolution

$$0 \to k[G] \to \bigoplus_{s \in S} k[G] \to k[G] \to k \to 0.$$

The image of $k[G]$ in $\bigoplus_{s\in S} k[G]$ is non-trivial, which implies that $\mathcal{D} \to \bigoplus_{s\in S} \mathcal{D}$ is injective. Hence, $\mathrm{H}_i(G;\mathcal{D})$ is only non-zero in degree one, and so $\dim_{\mathcal{D}} \mathrm{H}_1(G;\mathcal{D}) = |S| - 2$. Thus, knowledge of the Atiyah Conjecture for one-relator groups yields a very short computation of their L^2-Betti numbers (in the torsion-free case). While it is not known whether Hughes-free division rings exist for general torsion-free one-relator groups this holds whenever G is virtually compact special by Theorem 5.2.4. Many one-relator groups are known to be virtually compact special (see [Lin22]).

One-relator groups with torsion are virtually RFRS by [Wis21, Corollary 19.2], and therefore have a finite-index subgroup all of whose group algebras have Hughes-free embeddings. The computation of the k-L^2-Betti numbers of a Hughes-free embeddable subgroup of finite-index can be done using a length 2 projective resolution of $\mathbb{C}$ provided by the Lyndon Identity Theorem [Lyn50], and is not much more difficult than the computation above (see [DL07]). Thus $b_i^{(2)}(G) = 0$ for all $i > 1$ for G a one-relator group with torsion.

Example A.6 (Limit groups)**.** A finitely generated group G is a *limit group* if it has the same existential first order theory as some free group. Alternatively, G is a limit group if and only if it is fully residually free, meaning that for any finite subset $S \subseteq G$, there is a free group F and a homomorphism $G \to F$ that is injective on S. This definition immediately implies that limit groups are residually torsion-free nilpotent, and therefore $\mathcal{D}_{k[G]}$ exists for any field k and any limit group G.

By [Koc10, Corollary B], limit groups are free-by-(torsion-free nilpotent). So for any field k and limit group G, the Hughes-free division ring $\mathcal{D}_{k[G]}$ is of weak dimension at most one as a $k[G]$-module by Claim 4.3.12. It follows that

$$b_n^{(2)}(G;k) = \begin{cases} -\chi(G) & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this agrees with the homology gradient computations for limit groups carried out by Bridson and Kochloukova in [BK17, Corollary C] (this is a special case of the positive characteristic Lück Approximation Conjecture).

Example A.7 (Graphs of groups)**.** Let Γ be a graph and let $\mathcal{G} = (G_v, G_e; \Gamma)$ be a graph of groups with fundamental group $G = \pi_1(\mathcal{G})$. Suppose that G is locally indicable and that $\mathcal{D}_{k[G]}$ exists and assume that $b_n^{(2)}(G_e;k) = 0$ for every edge e of Γ and all $n \geqslant 0$. Then Chiswell's Mayer–Vietoris long exact sequence in group homology [Chi76] with coefficients in $\mathcal{D}_{k[G]}$ immediately implies that

$$b_n^{(2)}(G;k) = \sum_v b_n^{(2)}(G_v;k),$$

where the sum is taken over all the vertices v of Γ. A special and interesting case of this is when $\mathcal{G}$ is a graph of free groups with cyclic edge groups.

Let $G = \ast_{i\in I} G_i$, where each group G_i is non-trivial and locally indicable. If $\mathcal{D}_{k[G_i]}$ exists for each $i \in I$, then $\mathcal{D}_{k[G]}$ exists by [Sán08, Corollary 6.13(iv)]. Chiswell's long exact sequence then gives

$$b_n^{(2)}(G;k) = \begin{cases} \sum_{i\in I} b_n^{(2)}(G_i;k) + |I| - 1 & \text{if } n = 1 \\ \sum_{i\in I} b_n^{(2)}(G_i;k) & \text{otherwise.} \end{cases}$$

Example A.8 (3-manifold groups)**.** Let M be a 3-manifold with locally indicable fundamental group G. That G be locally indicable is not a strong assumption, as finitely generated 3-manifold groups are virtually locally indicable (see [AFW15, Flowchart 1]). Then $\mathcal{D}_{k[G]}$ exists by Theorem 5.3.7, so we can compute the k-L^2-Betti numbers of G. Note that the usual L^2-Betti numbers were first computed by Lott and Lück [LL95] before the resolution of Thurston's Virtual Fibring Conjecture.

By the Prime Decomposition Theorem (Theorem 5.3.4), it suffices to consider the case where M is irreducible with (possibly empty) incompressible boundary.

First assume that M has a non-empty boundary, and let $\Sigma_1, \dots, \Sigma_n$ be the incompressible boundary components of M; denote their fundamental groups by $H_1, \dots, H_n$. The collection of fundamental groups is denoted by $\mathcal{H}$ and the *homology of the pair* $(G, \mathcal{H})$ with coefficients in a $k[G]$-module M is given by

$$\mathrm{H}_i(G, \mathcal{H}; M) := \mathrm{Tor}_{i-1}^{k[G]}(M, \Delta_{G/\mathcal{H}}),$$

where $\Delta_{G/\mathcal{H}}$ is the kernel of the natural augmentation map $\bigoplus_{i=1}^n k[G/H_i] \to k$ (see [BE78]). The short exact sequence $0 \to \Delta_{G/\mathcal{H}} \to \bigoplus_{i=1}^n k[G/H_i] \to k \to 0$ induces a long exact sequence which contains the following portion:

$$\mathrm{H}_0(G, \mathcal{H}; \mathcal{D}_{k[G]}) \to \bigoplus_{i=1}^n \mathrm{H}_1(H_i; \mathcal{D}_{k[G]}) \to \mathrm{H}_1(G; \mathcal{D}_{k[G]}) \to \mathrm{H}_1(G, \mathcal{H}; \mathcal{D}_{k[G]}).$$

By definition, $\mathrm{H}_0(G, \mathcal{H}; \mathcal{D}_{k[G]}) = 0$. By Poincaré duality,

$$\mathrm{H}_1(G, \mathcal{H}; \mathcal{D}_{k[G]}) \cong \mathrm{H}^2(G; \mathcal{D}_{k[G]}).$$

But G is virtually free-by-cyclic by [KL23, Theorem 1.1], and therefore $b_2^{(2)}(G;k) = 0$ by Theorem 2.5.5. Then $\mathrm{H}^2(G; \mathcal{D}_{k[G]}) = 0$ by Proposition 2.5.4(iii). Hence, the k-L^2-Betti numbers of G are concentrated in degree one.

Now assume that M is a closed and irreducible. By the JSJ Decomposition Theorem 5.3.5 and Example A.7, the k-L^2-Betti numbers of G equal the sum of the corresponding k-L^2-Betti numbers of the individual pieces of the JSJ decomposition.

Each piece is either hyperbolic or Seifert fibred. The k-L^2-Betti numbers of the hyperbolic pieces all vanish by Example A.4, so it suffices to consider the Seifert fibred case. Let H be the fundamental group of a Seifert fibred manifold. By passing to a finite-index subgroup, we may assume that H fits into the (central) extension $1 \to \mathbb{Z} \to H \to \pi_1(\Sigma) \to 1$ for some surface Σ. The Lyndon–Hochschild–Serre spectral sequence associated to this extension is

$$\mathrm{H}_p(\pi_1(\Sigma); \mathrm{H}_q(\mathbb{Z}; \mathcal{D}_{k[H]})) \Longrightarrow \mathrm{H}_{p+q}(H; \mathcal{D}_{k[H]}).$$

But $\mathrm{H}_q(\mathbb{Z}; \mathcal{D}_{k[H]}) = 0$ for all $q \geqslant 0$ by Example A.1, and therefore $\mathrm{H}_{p+q}(H; \mathcal{D}_{k[H]}) = 0$.

Putting all of this together, we obtain the following statement: if M is an irreducible 3-manifold with non-trivial locally indicable fundamental group G and (possibly empty) incompressible boundary, then

$$b_n^{(2)}(G; k) = \begin{cases} -\chi(G) & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In a sense, the examples we have seen so far are not very interesting: the k-L^2-Betti numbers computed never depended on the ground field k and have always been concentrated in degree one. It is easy to construct examples where the latter property fails: $b_2^{(2)}(F_2 \times F_2; k) = 1$ for all fields k by a version of the Künneth formula. The next example–taken from the article [FHL24] of the author, Hughes, and Leary–generalises this, and gives examples of groups G where $b_n^{(2)}(G; k)$ depends on k.

Example A.9 (Right-angled Artin groups)**.** This example will be the most involved so far. Note that the classical L^2-Betti numbers of RAAGs were computed by Davis and Leary in [DL03].

The *right-angled Artin group* (RAAG) on the simplicial complex Γ is the group given by the presentation

$$A_\Gamma = \langle \{v\} \in \Gamma \mid [u, v] \text{ if and only if } \{u, v\} \in \Gamma \rangle.$$

Note that adding simplices of dimension at least two to Γ does not alter the presentation of A_Γ, so we will always assume that Γ is a flag complex.

We begin by recalling a Mayer–Vietoris spectral sequence constructed by Davis and Okun in [DO12]. Let P be a poset with the property that for every pair of elements $a, b \in \mathsf{P}$, there is a third element $c \in \mathsf{P}$ such that $c \leqslant a, b$. The *flag realisation* $\mathrm{Flag}(\mathsf{P})$ of P is the simplicial complex, whose simplices consist of finite chains of elements in P (we include empty chain). For example, if $\mathsf{P} = \{a < b\}$, then $\mathrm{Flag}(P)$ is a 2-simplex. For each simplex σ in $\mathrm{Flag}(\mathsf{P})$, let $\min(\sigma) \in \mathsf{P}$ be the minimal vertex in σ.

A *system of coefficients* on P is a covariant functor $\mathcal{A}\colon \mathsf{P} \to \mathsf{Mod}_R$ for some ring R. The homology $\mathrm{H}_\bullet(\mathsf{P};\mathcal{A})$ of P with coefficients in $\mathcal{A}$ is the homology of the chain complex

$$C_n(\mathrm{Flag}(\mathsf{P});\mathcal{A}) = \bigoplus_{\sigma\in\mathrm{Flag}(\mathsf{P})^{(n)}} \mathcal{A}(\min(\sigma))$$

where $\mathrm{Flag}(\mathsf{P})^{(n)}$ is the set of n-simplices in $\mathrm{Flag}(\mathsf{P})$, and the boundary maps are induced from the boundary maps of the simplicial chain complex of $\mathrm{Flag}(\mathsf{P})$ and the functor $\mathcal{A}$.

A *poset of spaces over* P is a CW complex X together with a set of subcomplexes $\mathcal{X} = \{X_a\}_{a\in\mathsf{P}}$ such that

(1) $X = \bigcup_{a\in\mathsf{P}} X_a$;
(2) $X_a \subset X_b$ whenever $a < b$;
(3) $\mathcal{X}$ is closed under non-empty finite intersections.

We are now ready to give the spectral sequence of [DO12]; we state the equivariant homological version.

Proposition A.9.1 ([DO12, Lemmas 2.1 and 2.2])**.** *Let $(X,\mathcal{X})$ be a poset of spaces over P. Let G be a group acting on X preserving each $X_a \in \mathcal{X}$. Let M be an arbitrary $k[G]$-module, and consider the systems of coefficients $\mathcal{H}_q^G$ on P given by $a \mapsto \mathrm{H}_q^G(X_a;M)$ for each $q \geqslant 0$. There is a convergent spectral sequence*

$$E^2_{p,q} = \mathrm{H}_p(\mathsf{P};\mathcal{H}_q^G) \Longrightarrow \mathrm{H}^G_{p+q}(X;M).$$

If the induced maps $\mathrm{H}_q^G(X_a;M) \to \mathrm{H}_q^G(X_b;M)$ are zero for all $q \geqslant 0$ and all pairs $a < b$, then the second page is given by

$$E^2_{p,q} = \bigoplus_{a\in\mathsf{P}} \mathrm{H}_p(\mathrm{Flag}(\mathsf{P}_{\geqslant a}),\mathrm{Flag}(\mathsf{P}_{>a});\mathrm{H}_q^G(X_a;M)),$$

where $\mathsf{P}_{>a}$ (resp. $\mathsf{P}_{\geqslant a}$) is the subposet of P consisting of elements greater than (resp. greater than or equal to) a.

Let A_Γ be the RAAG on the flag complex Γ and let X_Γ be the Salvetti complex of A_Γ. Let T_σ denote the combinatorial n-torus in X_Γ associated to the $(n-1)$-simplex σ of Γ. Hence, $T_\varnothing$ is the unique of X_Γ. Let $\widetilde{X_\Gamma}$ denote the universal cover of X_Γ, and let $\widetilde{T_\sigma}$ be the preimage of T_σ under the covering map $\widetilde{X_\Gamma} \to X_\Gamma$. The pair $(\widetilde{X_\Gamma},\{\widetilde{T_\sigma}\})$ is a poset of spaces over a poset P, where $\mathrm{Flag}(\mathsf{P})$ is isomorphic to the cone on the barycentric subdivision of Γ. Note that A_Γ acts on the poset of spaces, so we may apply Proposition A.9.1 (in the case $M = \mathcal{D}_{k[A_\Gamma]}$). By Example A.1,

$$\mathrm{H}_q^{A_\Gamma}(\widetilde{T_\sigma};\mathcal{D}_{k[A_\Gamma]}) \cong \mathrm{H}_q(\mathbb{Z}^{\dim(\sigma)+1};\mathcal{D}_{k[A_\Gamma]}) = 0$$

for all $q \geqslant 0$ whenever $\sigma \neq \varnothing$. Hence, the spectral sequence of Proposition A.9.1 collapses on page two, whose only non-zero terms have the form

$$E^2_{p,0} = \mathrm{H}_p\left(\mathrm{Flag}(\mathsf{P}), \mathrm{Flag}(\mathsf{P}_{>\varnothing}); \mathrm{H}_0^{A_\Gamma}(\widetilde{T_\varnothing}; \mathcal{D}_{k[A_\Gamma]})\right) = \mathrm{H}_p\left(\mathrm{Flag}(\mathsf{P}), \mathrm{Flag}(\mathsf{P}_{>\varnothing}); \mathcal{D}_{k[A_\Gamma]}\right).$$

Since $\mathrm{Flag}(\mathsf{P})$ is isomorphic to the cone $C\Gamma$ and $\mathrm{Flag}(\mathsf{P}_{>\varnothing})$ is isomorphic to Γ, the relative homology of the pair $(\mathrm{Flag}(\mathsf{P}), \mathrm{Flag}(\mathsf{P}_{>\varnothing}))$ is isomorphic to the homology of the suspension $S\Gamma$. Thus,

$$E^2_{p,0} \cong \mathrm{H}_p(S\Gamma; \mathcal{D}_{k[G]}) \cong \widetilde{\mathrm{H}}_{p-1}(\Gamma; \mathcal{D}_{k[G]}),$$

where $\widetilde{\mathrm{H}}_\bullet$ denotes reduced homology. We therefore obtain the following result, extending the computation of Davis and Leary, and showing that the k-L^2-Betti numbers of RAAGs depend on the ground field k.

Theorem A.9.2 ([DL03, Theorem 1],[FHL24, Theorem 3.15])**.** *Let A_Γ be the right-angled Artin group on the flag complex Γ. Then*

$$b_n^{(2)}(A_\Gamma; k) = \widetilde{b}_{n-1}(\Gamma; k)$$

for all $n \geqslant 0$.

We remark that the k-L^2-Betti numbers $b_n^{(2)}(A_\Gamma; k)$ agree with the homology gradients computed by Avramidi, Okun, and Schreve in [AOS21, Theorem 1].

To conclude, we remark that the k-L^2-Betti numbers have already shown to be useful obstruction to fibring. In [AOS24], Avramidi, Okun, and Schreve construct a 7-manifold M with Gromov hyperbolic fundamental group G such that $b_3^{(2)}(G; \mathbb{F}_p) > 0$ for odd primes p. Hence, M cannot virtually fibre over the circle. Note, however, that M is L^2-acyclic (and therefore is not a counterexample to the Singer Conjecture), so the usual L^2-Betti numbers do not suffice to obstruct virtual fibring. It is worth mentioning that G is virtually compact special, and therefore G virtually algebraically fibres with kernel of type $\mathrm{FP}(\mathbb{Q})$ by Theorem 3.2.3.

Appendix B

Questions and conjectures

B.1 Fibring

A natural direction of research is to promote the virtual algebraic fibrations Γ of simple type lattices in $\mathrm{SO}(n,1)$ (for n odd) to virtual fibrations of the space $\mathrm{SO}(n,1)/\Gamma$ over S^1. While this seems like a difficult problem, there are intermediate steps that could be more approachable, such as strengthening the finiteness properties of the kernels of the algebraic fibrations. The finiteness properties $\mathrm{FP}_n(k)$ for a field k are all strictly weaker than the finiteness property $\mathrm{FP}_n(\mathbb{Z})$, except when $n = 1$. It would thus be interesting if a homological invariant could detect virtual fibring with $\mathrm{FP}_n(\mathbb{Z})$ kernels among the class of RFRS groups. By Theorem A.9.2, there are RAAGs that are L^2-acyclic but not $\mathbb{F}_p$-L^2-acyclic for some primes p, and hence they can never fibre with kernel of type $\mathrm{FP}(\mathbb{Z})$. Hence L^2-Betti numbers alone do not detect virtual fibring in the class of RFRS groups. However, if a RAAG is $\mathbb{F}_p$-L^2-acyclic for all p, then it algebraically fibres with kernel of type $\mathrm{FP}(\mathbb{Z})$ (the Bestvina–Brady group is such a kernel in this case). This motivates the following more general conjecture.

Conjecture B.1. *Let G be a RFRS group of type $\mathrm{FP}_{n+1}(\mathbb{Z})$. The following are equivalent:*

1. *there exists a finite-index subgroup $H \leqslant G$ admitting an epimorphism $H \to \mathbb{Z}$ with kernel of type $\mathrm{FP}_n(\mathbb{Z})$;*
2. *$b_i^{(2)}(G;\mathbb{F}_p) = 0$ for all $i \leqslant n$ and for all primes p.*

It is known that being of type $\mathrm{FP}_2(k)$ over all fields k does not imply being of type $\mathrm{FP}_2(\mathbb{Z})$ by a result of R. Kropholler [Kro21], but the examples of such groups are not RFRS and it is not clear that they could occur as an algebraic fibre in a group of type $\mathrm{FP}(\mathbb{Z})$, so they are not counterexamples to Conjecture B.1. Note that if G is of type $\mathrm{FP}(\mathbb{Z})$ and is finitely presented, then G has a classifying space that is finitely dominated. Moreover, if G satisfies the Farrell–Jones Conjecture, then G admits a finite classifying space. By further work of Farrell [Far72], if $\pi_1(M)$ satisfies the Farrell–Jones Conjecture and M is a manifold of dimension at least 6, then algebraic fibrations with kernel of finite type are always induced by fibrations over S^1. Thus, if M is a manifold of dimension at least 7 with an $\mathbb{F}_p$-L^2-acyclic virtually compact special fundamental group for all primes p and Conjecture B.1 is true, then this would reduce the problem of virtually fibring M over the circle to exhibiting a virtual algebraic fibration of $\pi_1(M)$ with finitely presented kernel. While this is still a difficult task, an example of a hyperbolic 7-manifold with such an algebraic fibration has already been constructed [IMM24]. These questions are all very interesting even in the special case of uniform lattices of simple type in $\mathrm{SO}(n,1)$, which are already known to be virtually compact special [BHW11].

Conjecture B.2. *Let $\Gamma < \mathrm{SO}(n,1)$ be a lattice. For all primes p, the $\mathbb{F}_p$-L^2-Betti numbers of Γ are concentrated in the middle dimension.*

Using the terminology of [AOS24], an alternate way to state this conjecture is that such a lattice Γ satisfies the $\mathbb{F}_p$-Singer Conjecture for all primes p. This is closely related to the problem of determining the mod p homology growth of lattices $\Gamma < \mathrm{SO}(n,1)$. While there are vanishing results for mod p homology growth of lattices of higher rank [ABFG25], very little is known in rank one.

B.2 Coherence and virtually free-by-cyclic groups

While it seems unlikely that Conjecture 1.2.5 is true in full generality–especially the implication from coherence to the vanishing of the second L^2-Betti number–it is very plausibly true in some well-behaved classes of groups.

Question B.3. *Are homological coherence and the vanishing of the second L^2-Betti number equivalent in the class of RFRS groups of rational cohomological dimension at most two?*

A positive answer to Question B.3 would imply that RFRS groups of cohomological dimension at most two are coherent if and only if they are virtually free-by-cyclic, and it would also imply that homological coherence is equivalent to coherence for 2-dimensional RFRS groups.

Wise conjectures that all hyperbolic one-relator groups are virtually free-by-cyclic [Wis20a, Conjecture 17.8]. We believe it is likely the case that all hyperbolic one-relator groups are virtually special, so Wise's conjecture would follow from [KL24] or

Theorem 1.2.1. In fact we propose the following characterisation of virtually free-by-cyclic one-relator groups.

Conjecture B.2.1. *A one-relator group is virtually free-by-cyclic if and only if it is virtually special.*

In Corollary 4.3.15, it was shown that Strong Parafree Conjecture implies that finitely generated parafree groups are coherent. This motivates the following weaker conjecture, which will be familiar to experts but which the author has not seen appear in the literature.

Conjecture B.4. *Finitely generated parafree groups are coherent.*

By Corollary 1.2.4, poly-$\mathbb{Z}$ groups of cohomological dimension at most two with vanishing second L^2-Betti number are coherent and have coherent group algebras. In fact, many classes of coherent groups are also known to have coherent group algebras, such as the classes of free-by-cyclic groups [FH99, HLÁ22], one-relator groups [JZL23], and coherent elementary amenable groups [HKKL24]. In general, neither implication in the following question is known (recall that we require all finitely generated one-sided ideals to be finitely presented for a ring to be coherent).

Question B.5. *Let G be a group and let k be a field. Is G coherent if and only if $k[G]$ is coherent?*

The following special case is of historical interest, since 3-manifold groups have long been known to be coherent by Scott's Core Theorem [Sco73]. A solution would like involve pushing current methods beyond cohomological dimension two.

Conjecture B.6. *If M is a 3-manifold, then $k[\pi_1(M)]$ is coherent for any field k.*

By the JSJ and Prime Decomposition Theorems, it would suffice to prove the conjecture in the case that M is a hyperbolic 3-manifold or a Seifert fibred space.

The methods used in Chapter 4 and in [JZL23] to prove that a group algebra is coherent use Hughes-free embeddings. There is not a good notion of a Hughes-free embedding for the group ring $\mathbb{Z}[G]$ with integral coefficients (or other non-division ring coefficients), and thus it is often not possible to prove that $\mathbb{Z}[G]$ is coherent.

Question B.7. *Is there a group G such that $\mathbb{Q}[G]$ is coherent but $\mathbb{Z}[G]$ is incoherent?*

The following special cases are interesting.

Conjecture B.8. *$\mathbb{Z}[G]$ is coherent if G is either a free-by-cyclic group or a one-relator group.*

B.3 Division rings

In Section 5.1, it is shown that if $\mathcal{G} = (G_v, G_e)$ is a graph of groups and k is a field such that $k[G_v]$ admits a Hughes-free embedding for each vertex group G_v, then $k[\pi_1(\mathcal{G})]$ admits an embedding into a division ring. One can ask several questions about the properties of this embedding.

Question B.9. *Let $\mathcal{G} = (G_v, G_e)$ be a graph of groups, let k be a field, and suppose that $k[G_v]$ admits a Hughes-free embedding. Let $k[\pi_1(\mathcal{G})] \hookrightarrow \mathcal{D}$ be the embedding constructed in Theorem 5.1.10.*

(1) *Is the embedding Linnell?*
(2) *If $\pi_1(\mathcal{G})$ is locally indicable, is the embedding Hughes-free?*
(3) *If $\pi_1(\mathcal{G})$ is locally indicable, is the embedding universal?*

A positive answer to (1) of course implies a positive answer to (2). Using the Magnus hierarchy for one-relator groups, (2) and (3) also imply the following conjecture, which is an important special case of a conjecture of Jaikin-Zapirain [JZ21, Conjecture 1].

Conjecture B.10. *If G is a torsion-free one-relator group, the $k[G]$ admits a Hughes-free and universal embedding into a division ring for every field k.*

When k is of characteristic zero, then it is known that a Hughes-free embedding exists by the resolution of the Atiyah Conjecture for locally indicable groups [JZLÁ20]. It is still unknown, however, whether this embedding is universal. In the case of 3-manifold groups, we could show that if $\pi_1(M^3)$ is locally indicable, then its group algebras admit Hughes-free embeddings.

Question B.11. *If $\pi_1(M^3)$ is locally indicable, are the embeddings of its group algebra into Hughes-free division rings constructed in Theorem 5.3.8 universal?*

This would follow easily from a positive answer to the following question, which is yet again an interesting special case of Jaikin-Zapirain's conjecture that should be more tractable.

Question B.12. *Let G be a locally indicable group that is virtually compact special. Are the Hughes-free embeddings of its group algebras constructed in Theorem 5.2.4 universal?*

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